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Legendre and Chebyshev Polynomials for Solving Mixed Integral Equation

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Abstract

In this paper, the solution of mixed integral equation (MIE) of the first and second kind in time and position is discussed and obtained in the space $L_2[-1,1] \times C[0,T], T < 1$. The kernel of position is established in the logarithmic form, while the kernels of time are continuous and positive functions in C[0,T]. A numerical method is used to obtain a linear system of Fredholm integral equations (SFIEs). In addition, the solution FIE of the second kind, with singular kernel, is solved, using Legendre polynomials. Moreover, Orthogonal polynomials methods are used to obtain the solution of singular FIE of the first kind.

Keywords: *Mixed integral equation; contact problem; Legendre polynomial; Krein's method; Chebyshev polynomial.*

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1 Introduction

The mathematical physics and contact problems in the theory of elasticity lead to an integral equation of the first or second kind, see [1,2,3]. Mkhitarian and Abdou [4,5] discussed some different methods for solving the **FIE** of the first kind**,** with logarithmic kernel [4], and Carleman kernel [5] respectively. More information for solving the integral equations and the fractional partial differential equations with its applications can be found in [6-9].

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In this work, we consider the **MIE**

$$\int_0^t \int_{-1}^1 F(t,\tau)k\left(\frac{x-y}{\lambda}\right)\varphi(y,\tau)dy\,d\tau + \int_0^t G(t,\tau)\varphi(x,\tau)d\tau = [\gamma(t) - f_*(x)] = f(x,t), \qquad (1.1)$$

Where

$$k(v) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\tanh u}{u} e^{iuv} du \quad , i = \sqrt{-1},$$

under the condition

$$\int_{-1}^1 \varphi(x,t)dx = P(t) \qquad (1.2)$$

The two given functions $F(t,\tau)$ and $G(t,\tau)$, for $t \in [0,T], T < 1$ are positive, continuous with its derivatives in the class $C[0,T]$, and represent the two kernels of Volterra integral term. The bad function $k\left(\frac{x-y}{\lambda}\right), \lambda \in (0,\infty)$ is called the kernel of Fredholm integral term, in the domain $[-1,1]$. The given continuous functions $\gamma(t)$ and $f(x)$ belongs, respectively to the class $C[0,T]$, and the space $L_2[-1,1]$. The given function $f(x,t)$ is continuous with its partial derivatives. The unknown function $\varphi(x,t)$ will be obtained in the space $L_2[-1,1] \times C[0,T], T < 1$ The integral equation (1.1), under the condition (1.2), is investigated from the contact problem of a rigid surface (G,ν) having an elastic material, where G is the displacement magnitude and ν is Poisson's coefficient. If a stamp of length 2 unit and its surface is describing by the formula $f_*(x)$, is impressed into an elastic layer surface of a strip by a variable force $P(t), 0 \le t \le T < 1$, whose eccentricity of application $e(t)$, that cases rigid displacement $\gamma(t)$. Here the function $F(t,\tau)$ represents the resistance force of material in the domain of contact $[-1,1]$, through the time $t \in [0,T]$ and $G(t,\tau)$ is the external force that supplied through the domain of contact problem to increase the resistance of the domain. As in Ref. [10], the kernel of position of (1.1) can be written in the form

$$k(v) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\tanh u}{u} e^{iuv} du = -\ln\left|\tanh\frac{\pi v}{4}\right| , v = \left(\frac{x-y}{\lambda}\right) , \lambda \in (0,\infty). \qquad (1.3)$$

If $\lambda \to \infty$ and $(x-y)$ is very small, so that the condition $\tanh z \approx z$, then we have

$$ln\left|tanh\frac{\pi v}{4}\right| = ln|v| - d \,, d = ln\frac{4\lambda}{\pi} \,, d = ln\frac{4\lambda}{\pi} \qquad (1.4)$$

Hence, the formula (1.1) becomes

$$-\int_0^t \int_{-1}^1 F(t,\tau)[\ln|x-y| - d]\varphi(y,\tau)dy\,d\tau + \int_0^t G(t,\tau)\varphi(x,\tau)d\tau = f(x,t), \qquad (1.5)$$

In order to guarantee the existence of a unique solution of Eq. (1.1) or (1.5), under the condition (1.2), we assume the following conditions:

(i) The kernel of position $k(|x-y|)$ satisfies the discontinuous condition
$\left\{\int_{-1}^1 \int_{-1}^1 k^2\left(\left|\frac{x-y}{\lambda}\right|\right) dx\,dy\right\}^{1/2} < A$, A is a constant.

(ii) For all values of $t,\tau \in [0,T]$ the two functions $F(t,\tau)$, $G(t,\tau)$ with its derivatives belong to For all values of $t,\tau \in [0,T]$ the two functions (t,τ), $G(t,\tau)$ $|F(t,\tau)| < B$; $|G(t,\tau)| < D$, for all $C([0,T] \times [0,T])$ and satisfy the following conditions. values of $\tau \in [0,T]$, where B and D are constants.

(iii) The function $f(x,t) \in L_2[-1,1] \times C[0,T]$ and its normality in $L_2[-1,1] \times C[0,T]$ is defined as
$\|\emptyset(x,t)\| = max_{0 \le t \le T} \int_0^t \left\{\int_{-1}^1 f^2(x,\tau)dx\right\}^{1/2} d\tau$

(iv) The unknown function $\varphi(x,t)$ satisfies Hölder condition with respect to time and Lipschitz condition with respect to position.

In this work, a numerical method is used to obtain **SFIEs** of the first kind or of the second kind according to on the relation between the derivatives of the two functions $F(t,\tau)$ and $G(t,\tau)$, for all the values of $\tau \in [0,T]$, with respect to the time t. In section 3, we represent the unknown function in the Legendre polynomial form. In addition, in section 4, orthogonal polynomials method is used to discuss the solution of **FIE** of the first kind in the form of spectral relationships. The stability of the solution is discussed. In section 5, numerical results and general conclusions with many important cases are considered and discussed.

2 Numerical Methods

To discuss the solution of (1.5), under (1.2), we divide the interval $[0,T], 0 \le t \le T < 1$ as $0 = t_0 < t_1 < \cdots < t_N = T$; $0 =< t = t_k$, $k = 1,2,3,\ldots,N$, to get

$$-\int_0^{t_k}\int_{-1}^{1} F(t_k,\tau)(\ln|x-y|-d)\varphi(y,\tau)dy\,d\tau + \int_0^{t_k} G(t_k,\tau)\varphi(x,\tau)d\tau = f(x,t_k), \quad (2.1)$$

$$\int_{-1}^{1}\varphi(x,t_k)dx = P(t_k) \quad (2.2)$$

Hence, we have

$$-\sum_{j=0}^{k} u_j F(t_k,t_j)\int_{-1}^{1}(\ln|x-y|-d)\varphi(y,t_j)dy + \sum_{j=0}^{k} u_j G(t_k,t_j)\varphi(x,t_j) + O(\hbar_j^{P+1}) = f(x,t_k) \quad (2.3)$$

Where $\hbar_j = \max_{0\le j\le k} h_j$, $\hbar_i = \max_{0\le i\le k} h_i$, $h_1 = t_{l+1} - t_l$..

The values of u_j, P, v_i and $\tilde{P}$ are depending on the number of the derivatives of $F(t,\tau)$ and $G(t,\tau)$, for $\tau \in [0,T]$, with respect to t. For example, if $F(t,\tau) \in C^4[0,T]$, then we have $P = 4, k \cong 4$ in the first integral term of Eq. (2.3), where we get $u_0 = \frac{h_0}{2}$, $u_4 = \frac{h_4}{2}$; $u_n \doteq h_n$; $n = 1,2,3$. and $u_n = 0$, for $n > 4$. While, if $G(t,\tau) \in C^3[0,T]$ then we have $\tilde{P} = 3, k \cong 3$ for the second term of (2.3), hence $v_0 = \frac{h_0}{2}, v_3 = \frac{h_3}{2}$; $u_n = h_n; n = 1,2$, $m = 1,2$ and $v_n = 0$ for $m > 3$. More information for the characteristic points and the quadrature coefficients are found in [11,12]. Using the following notations

$$F(t_k,t_j) = F_{k,j}\ , G(t_k,t_i) = G_{k,i}\ , \varphi(y,t_n) = \varphi_n(y)\ , f(x,t_m) = f_m(x) \quad (2.4)$$

$(j,i,n,m = 0,1,\ldots,k\ ; 0 < k \le N)$. The formula (2.4) rewrite in the form

$$-\sum_{j=0}^{k} u_j F_{k,j}\int_{-1}^{1}(\ln|x-y|-d)\varphi_j(y)dy + \sum_{j=0}^{k} u_j G_{k,j}\varphi_j(x) = f_k(x) \quad (2.5)$$

In addition, the boundary condition (2.2) becomes

$$\int_{-1}^{1}\varphi_k(x)dx = P_k\ , 0 < k \le N\ ;\ (P_k \quad Cons.). \quad (2.6)$$

Now, we have the following discussion

(1) The formula (2.5) represents a linear system of FIE of the second kind, for all cases when the two functions $F(t,\tau)$and $G(t,\tau)$ have the same derivatives with respect to time $t \in [0,T]$. (2) When the function G(t,τ) has n derivatives such that $n < k$ the formula (2.5), in this case, takes the forms

$$-\sum_{j=0}^{n} u_j F_{n,j} \int_{-1}^{1} (\ln|x-y| - d)\varphi_j(y)dy + \sum_{j=0}^{n} u_j G_{n,j} \varphi_j(x) = f_n(x),$$
$$(n < k, k = 1,2, \dots N), \tag{2.7}$$

and

$$-\sum_{j=n+1}^{k} u_j F_{k,j} \int_{-1}^{1} (\ln|x-y| - d)\varphi_j(y)dy = f_k(x) - \sum_{j=0}^{k} \mu_j\big(u_j, G_{n,j}, F_{n,j}\big)\varphi_j(x) \tag{2.8}$$

Hence, the formula (2.7) represents linear S**FIEs** of the second kind, which can solve using the recurrence relations. After obtaining the solution of the system (2.7), we can obtain the solution of Eq. (2.8), which represents linear **SFIEs** of the first kind.

(2) When the function $F(t, \tau)$ has n derivatives such that $n < k$, hence the formula (2.5) leads to Eq. (2.7) and the following algebraic system

$$-\sum_{j=n+1}^{k} \nu_j G_{k,j} \varphi_j(x) = f_k(x) - \sum_{j=0}^{n} \beta_j\big(u_j, G_{n,j}, F_{n,j}\big)\varphi_j(x)\,; \tag{2.9}$$

where β_j are constants

3 Fredholm Integral Equation of the Second Kind

To obtain the solution of Eq. (2.7), we adapt it in the form

$$\mu_k \varphi_k(x) - \mu_k' \int_{-1}^{1} (\ln|x-y| - d)\varphi_k(y)dy = \sum_{j=0}^{k-1} u_j F_{k,j} \int_{-1}^{1} (\ln|x-y| - d)\varphi_j(y)dy$$

$$= f_k(x) - \sum_{j=0}^{k-1} \nu_j G_{k,j} \varphi_j(x)\,;\ \left(\mu_k = \frac{h_k}{2} G_{k,k}\,, \mu_k' = \frac{h_k}{2} F_{k,k}\,, G_{k,k} \neq 0, F_{k,k} \neq 0\right) \tag{3.1}$$

The solution of (3.1) can be obtained using the recurrence relation, for this let $k = 0$ in (3.1) to get

$$\mu_0 \varphi_0(x) - \mu_0' \int_{-1}^{1} (\ln|x-y| - d)\varphi_0(y)dy = f_0(x) \tag{3.2}$$

To obtain the solution of (3.2), we assume the unknown function $\varphi_0(x)$ in the Legendre polynomials form

$$\varphi_0(x) = \sum_{n=0}^{\infty} C_n^{(0)} P_n(x) \tag{3.3}$$

where $C_n^{(0)}$ are constants and $P_n(x)$ are the Legendre polynomials that satisfy the orthogonal relation [13]

$$\int_{-1}^{1} P_n(x) P_m(x) dx = \begin{cases} \frac{2}{2n+1} & n = m \\ 0 & n \neq m \end{cases} \tag{3.4}$$

The polynomial series (3.3), at the two end points of contact $x = \pm 1$, behaves as $\varphi_0(-1) = \sum_{n=0}^{\infty} (-1)^n C_n^{(0)}$. Also, we say that, if $\varphi_0(x) \in L_2[-1,1]$ then the polynomial series (3.3) belongs to $L_2[-1,1]$, see [13]. In view of (3.3), we differentiate (3.2) with respect to x, hence we obtain

$$\delta_0 \varphi_0' - \int_{-1}^{1} \frac{\varphi_0(y)dy}{x-y} = g_0\,;\ \left(\delta_0 = \frac{\mu_0}{\mu_0'}\,, g_0(x) = \frac{f'(x)}{\mu_0'}\right) \tag{3.5}$$

In addition, from (3.3), we have, see [13]

$$\frac{d}{dx}\varphi_0(x) = \sum_{n=0}^{\infty} C_n^0 P_n'(x).(1-x^2)^{-\frac{1}{2}} \tag{3.6}$$

Here, $P_n^m(x), n, m \geq 0$ are the associated Legendre polynomials of the first kind, that satisfy the following general orthogonal relation, see [14.p.808]

$$\int_{-1}^{1} P_n^l(x) P_m^l(x) dx = \begin{cases} \frac{2(n+l)!}{(n-k)!(2n+l)} & n = m \\ 0 & n \neq m \end{cases} \tag{3.7}$$

In view of Eq. (3.3) and Eq.(3.6), the known function of Eq. (3.5) can be represented as, see [9].

$$g_0(x) = -\sum_{n=0}^{\infty} \tilde{g}_n^{(0)} P_n'(x).(1-x^2)^{-\frac{1}{2}} \tag{3.8}$$

the coefficients $g_n^{(0)}, n \geq 0$ are constants, which can be determined after using Eq. (3.7). When $g_0 \in L_2[-1,1]$, it follows that, the polynomial series (3.8) belongs to $L_2[-1,1]$.

Using the following relation, (see [14], p.835)

$$Q_n(x) = \frac{1}{2}\int_{-1}^{1} \frac{P_n(y)dy}{x-y} \tag{3.9}$$

The integral equation (3.5), with the aid of (3.6), (3.8) and (3.9), becomes

$$2\sqrt{1-x^2}\sum_{n=0}^{\infty} C_n^{(0)} Q_n(x) = \sum_{n=0}^{\infty}\left(\tilde{g}_n^{(0)} + \delta_0 C_n^{(0)}\right) P_n'(x) \tag{3.10}$$

$Q_n^l(x), n, l \geq 0$ are the associated Legendre functions of the second kind that satisfy the relations (see [14], Eq.7112, p.807;p.808)

$$\int_{-1}^{1} Q_n^l(x) P_m^l(x) dx = \frac{(-1)^l[1-(-1)^{n+m}](n+l)!}{(m-n)(m+n+l)(n-l)!}, l \geq 0 \tag{3.11}$$

and

$$\int_{-1}^{1}\sqrt{1-x^2}\, Q_n(x) P_m'(x) dx = \begin{cases} \frac{-2n(m+1)[1+(-1)^{n+m}]}{(m-n-1)(m-n+1)(m+n)(m+n+2)} & n \neq m \pm 1 \\ 0 & n = m \pm 1 \end{cases} \tag{3.12}$$

Multiplying both sides of (3.10) by the term $P_n' dx$, then integrating the result from -1 to 1, and using Eq. (3.12), we have ,

$$\delta_0 C_m^{(0)} + 2\sum_{n=0}^{\infty} \frac{2n(m+1)[1+(-1)^{n+m}]C_n^{(0)}}{(m-n-1)(m-n+1)(m+n)(m+n+2)} = g_m^{(0)}, \left(C_m^{(0)} = \frac{P_0}{2}; m \geq 1\right) \tag{3.13}$$

Following the same previous way, and using the mathematical induction, we can obtain the following relation:

$$\delta_k C_m^{(k)} + 2\sum_{n=0}^{\infty} \frac{(2m+1)[1+(-1)^{n+m}]C_n^{(k)}}{(m-n-1)(m-n+1)(m+n)(m+n+2)} = g_m^{(k)} + \sum_{j=0}^{k-1} u_j F_{j,k} C_j^{(k)};$$

$$\left(C_m^{(k)} = \frac{P_k}{2}; k = 0, 1, \ldots, N\right) \tag{3.14}$$

Abdou, in [15], proved that the infinite system of the linear algebraic Eq. (3.13) is regular for all values of δ_k , $0 \le k \le N$, that must satisfy the inequality $\min_{k>0}|\delta_k| > \frac{11}{5}$. Following the same way of Abdou [15], we can write Eq. (3.14) in the form of even functions;

$$\delta_{2k} X_{2m}^{(2k)} + \sum_{n=1}^{\infty} b_{2m,2n} X_{2n}^{(2k)} = H_{2m}^{(2k)} - \frac{P_{2k}}{2} b_{2m,0} \,, \tag{3.15}$$

$$H_{2m}^{(2k)} = g_{2m}^{(2k)} + \sum_{j=0}^{2k-2} u_{2j} F_{2j,2k} C_{2m}^{(2k)} \ , k = 0, 1, \dots, N/2, \tag{3.16}$$

for odd functions

$$\delta_{(2k-1)} X_{(2m-1)}^{(2k-1)} + \sum_{n=1}^{\infty} b_{(2m-1,2n-1)} X_{(2n-1)}^{(2k-1)} = H_{(2m-1)}^{(2k-1)} - \frac{3}{2} M_{(2k-1)} b_{(2m-1,0)}, \tag{3.17}$$

$$H_{(2m-1)}^{(2k-1)} = g_{(2m-1)}^{(2k-1)} + \sum_{j=0}^{2k-1} u_{(2j-1)} F_{(2j-1,2k-1)} C_{(2m-1)}^{(2k-1)}, k = 0, 1, \dots, \frac{N}{2}. \tag{3.18}$$

The regularity of the infinite system can be discussed, by following [15].

4 Fredholm Integral Equation of the First Kind

In this section, the solution of the linear system of **FIE** the first kind will be obtained using orthogonal polynomials method. This method has large applications in the theory of elasticity and contact problems.

Consider the **FIE** of the first kind in the form

$$\mu \int_{-1}^{1} (-\ln|x-y| + d) \psi(y) dy = g(x) \tag{4.1}$$

under static condition

$$\int_{-1}^{1} \psi(y) dy = L, \ L \text{ is a constant} \tag{4.2}$$

Theorem 1: The spectral relationships of the **FIE** of the first kind with logarithmic kernel, and when the free term is represented in a Chebyshev polynomial form for even function, takes the form

$$\int_{-1}^{1} (-\ln|x-y| + d) \frac{T_{2n}(y)}{\sqrt{1-y^2}} dy = \begin{cases} (\ln 2 + d)^{-1} & n = 0 \\ \frac{T_{2n}(x)}{2n} & n = 1, 2, \dots \end{cases} \tag{4.3}$$

and for odd function , takes the form

$$\int_{-1}^{1} (-\ln|x-y| + d) \frac{T_{2n-1}(y)}{\sqrt{1-y^2}} dy = \frac{T_{2n-1}(x)}{2n-1}, \ n \ge 1 \tag{4.4}$$

Where $T_n(x)$ is the Chebyshev polynomial of the first kind.

For the Chebyshev polynomials, one assumes $T_n(x) = cos(n\, cos^{-1} x)$, $x \in [-1,1]$, $n \ge 0$ as the Chebyshev polynomials of the first kind, while $U_n(x) = \frac{\sin[(n+1)\cos^{-1} x]}{\sin(\cos^{-1} x)}$, $n \ge 0$, the Chebyshev polynomials of the second kind. We know that $\{T_n(x)\}$ an orthogonal sequence of functions with respect to the weight function $(1-x^2)^{-\frac{1}{2}}$, while $\{U_n(x)\}$ form an orthogonal sequence of functions with respect to the weight function $(1-x^2)^{\frac{1}{2}}$. It appears reasonable to attempt a series expansion to $\emptyset(x)$ in Eq. (4.1) in terms of Chebyshev polynomials of the first kind. This choice is not arbitrary since one can identity a portion of the integral as the weight function associated with $T_n(x)$.

For convenience, we use the orthogonal polynomials method with some known algebraic and integral relations associated with Chebyshev polynomials see [16,17]. Thus, in this aim, we represent $\emptyset(x)$, $g(x)$ in the following forms

$$\emptyset(x) = \frac{1}{\sqrt{1-x^2}} \sum_{n=0}^{\infty} a_n T_n(x)\,, g(x) = \sum_{n=0}^{\infty} \frac{g_n T_n(x)}{\sqrt{1-x^2}}\,, \tag{4.5}$$

Using the above expressions of (4.5) in (4.1), hence Theorem 1is completely proved.

Differentiating (4.3) with respect to *x*, we get

$$\int_{-1}^{1} \frac{T_n(y)\, dy}{(y-x)\sqrt{1-y^2}} = \pi U_{n-1}(x)\,, n \geq 1. \tag{4.6}$$

The formula (4.6) represents spectral relationships of **FIE** of the first kind by Chebyshev of the second kind.

5 Numerical Results

Example (1): The solution of the IE (3.2) depends on the Cauchy, and the value of the given function $f(x)$. Here, we assume the following: $P_1(x) = x$, $c = 0.8$, and $N = 20$ and we use the general assumption of the solution $\varphi(x) = \sum_{j=0}^{20} a_j P_j(x)$, where $P_j(x)$ is Legendre polynomial and the coefficients a_j are the solution of the linear system (4.5), we used Maple (12) to solve such system. These coefficients are tabulated in Table 1.

Table 1. The values of Legendre polynomial series from $n = 1$ to $n = 20$

0.79651	1.28296	-1.39797	1.45698	-1.54907
1.47988	-1.56738	1.41765	-1.55909	1.17834
-1.39227	2.03712	-1.59626	-2.92476	-5.24833
8.38416	6.33867	1.70470	-0.25283	-1.19672

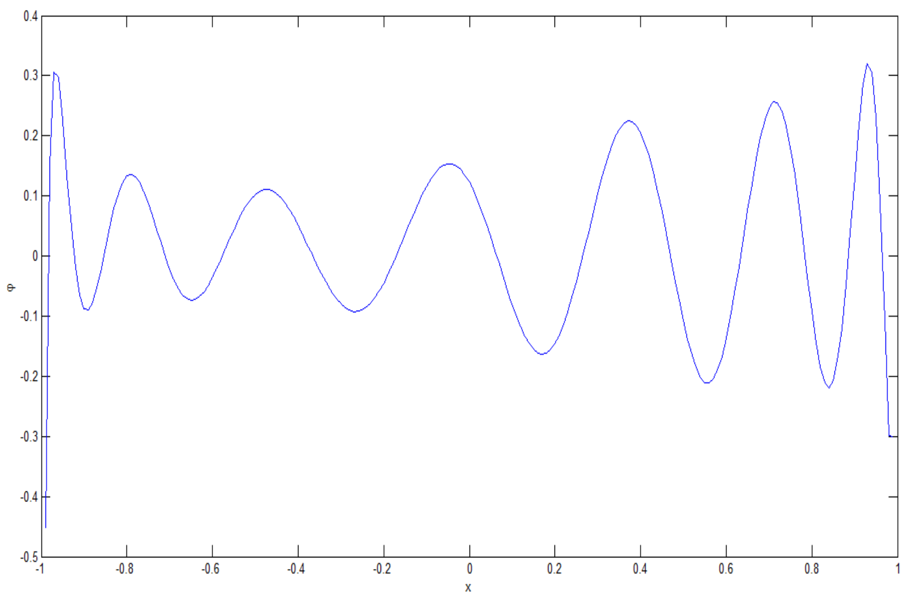

Fig. 1. Values of $\emptyset(x)$ for Legendre polynomial to 20 terms

Example (2): The behavior of the solution $\varphi_l(x)$ which is represented numerically by (4.1) and its solution, when $l = 0$ is given by Eq. (3.3). The general behavior can be described in Fig. 1 when the constant $\mu_0 = \frac{1}{2}, d = \frac{1}{2}, f(0,0) = \frac{1}{2}$ and the portion points $M = 10$. In Fig. 2 for the same values of μ, d, f and $M = 50$, we have the approximate solution of $\varphi_0(x)$.

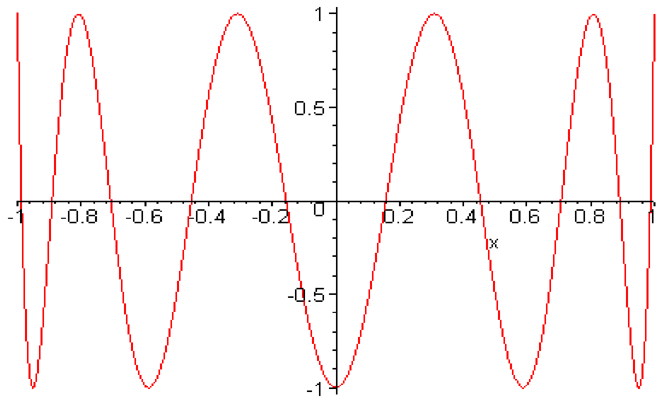

Fig. 2. Values of $\emptyset(x)$ for Legender polynomial to 50 terms

6 Conclusion

From the above results and discussion, the following may be conclude.

1. The mixed type of integral equation with Carleman kernel can be established from this work, by using the following famous relation see [17],

$$\ln|x-y| = H(x,y)|x-y|^{-\alpha} \qquad 0 \leq \propto < 1 \tag{5.1}$$

Where $H(x,y) = |x-y|^{\propto} \ln|x-y| \in C[-1,1]$. The importance of Carleman function came from the work of Artiurian [18], has shown that the plane contact problems of the nonlinear theory of plasticity, in its first approximation, reduce to **FIE** of the first kind with Carleman.

2. The formula (3.5), after using the transformations $y = 2u - 1, x = 2v - 1$ we obtain the following integral equation,

$$\frac{d\Theta}{dv} - \lambda' \int_0^1 \frac{\Theta(u)du}{v-u} = z(v) \tag{5.2}$$

This equation has appeared in both combined infra-red gaseous radiations and molecular conduction. λ', in Eq. (5.2), is the radiations conduction number for the large path length limit, represents the single parameter of the dimensionless system. Under the conditions $\Theta(\pm 1) = 0$, $z(u) = \frac{1}{2} - u$. The solution is obtained and discussed by Frankel [19].

3. For an infinite rigid strip with $2a$ impressed in a layer of viscous liquid of thickness for $V = V_0 e^{-iwt}$ where V_0 is a constant and w is the angular velocity resulting from rotating the strip about z-axis, the integral equation of such problem takes the form $\int_{-1}^{1} \tau(y) \ln \left|\tanh \frac{\pi(y-x)}{4h}\right| dy = \pi\rho V_0$

Where ρ the velocity coefficient.

4. The function $\left(\frac{\tanh u}{u}\right)$ is called the symbol kernel of position of the integral Eq. (1.1). If we present y as a complex variable $y = u + iv$ in (1.1), we easily see that the symbol kernel satisfies the regular analytic function in the strip $|u| < \infty$.
5. The kernel of position can be written in the following forms

(i) Weber-Sonien formula

$$\ln|u - v| = \sqrt{uv}\int_0^\infty J_{\pm\frac{1}{2}}(wu)J_{\pm\frac{1}{2}}(wv)dw$$

where, $\pm\frac{1}{2}$ for symmetric and skew-symmetric kernel, respectively, and $J_n(x)$ is the Bessel function of the first kind.

(ii) The Legendre polynomial formula

$$\ln|u - v| = \left(\frac{uv}{2}\right)\sum_{n=0}^{\infty}\frac{\Gamma(n+1)P_n^{\frac{1}{2}}(u)P_n^{\frac{1}{2}}(v)}{\Gamma^2\left(\frac{3}{2}+n\right)(1+2n)^{-1}} \qquad \text{(Skew- symmetric problem)}$$

$$\ln|u - v| = \sum_{n=0}^{\infty}\frac{2n\Gamma(n)P_n^{\frac{1}{2}}(u)P_n^{\frac{1}{2}}(v)}{\Gamma^2\left(\frac{1}{2}+n\right)} \qquad \text{(Symmetric problem)}$$

when $\Gamma(n)$ is the Gamma function and $P_n^m(x)$ is the associated Legendre polynomial of the first kind.

6. The contact problem of a rigid surface having an elastic material, when a stamp of length 2 unit is impressed into an elastic layer surface of a strip by a variable force $p(t)$, $0 \le t \le T < 1$, whose eccentricity of application $e(t)$ represents Fredholm – Volterra integral equation.
7. The numerical method used enables us to obtain the solution of the system of Fredholm equations by the recurrence relations.
8. The kind of the system of Fredholm equation depends of the resistance force of material $F(t, \tau)$ and the external resistance force $G(t, \tau)$.
9. The three kinds of the displacement problems, in the theory of elasticity and contact problem, which included and discussed in [20], are consider special case of this work.

Competing Interests

Authors have declared that no competing interests exist.

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