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A New Spectral-Collocation Method Using Legendre Multi-wavelets for Solving of Nonlinear Fractional Differential Equations

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

In this paper, a novel spectral collocation method using Legendre multi-wavelets as the basis functions is presented to obtain the numerical solution of nonlinear fractional differential equations. The fractional derivative is described in the Caputo sense. The two-scale relations of Legendre multi-wavelets and the properties of block pulse functions have been used in the evaluation of the fractional integral operational matrix and expansion coefficients of the nonlinear terms for the Legendre multi-wavelets. Due to the aforementioned properties, the original differential equation is converted into a nonlinear system of algebraic equations which can be solved by existing tools. The numerical results are compared with exact solutions and existing numerical solutions found in the literature and demonstrate the validity and applicability of the proposed method.

Keywords: *Legendre multi-wavelets; fractional integral operational matrix; two-scale relations; fractional differential equation.*

2010 Mathematics Subject Classification: 35R11, 65M06, 65M12.

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1 Introduction

Fractional differential equation is a very effectively tool for the description of memory and hereditary properties in various materials and processes [1-4], such as gas diffusion and heat conduction in fractal porous media. Although a lot of attentions have been paid on fractional calculus and a large literature for solving fractional differential equation exists, there remain several challenges, and therefore, it is necessary to improve the current methods or develop new ones. Among the issues are as follows: (i) how to accurately represent fractional integral or differential operator, (ii) how to effectively deal with high nonlinearity and (iii) how to significantly reduce the computational cost.

In the past two decades there has been a considerable interest to the wavelets methods in the numerical solution of fractional differential equations. Wavelets possess orthogonality, compactly supported, and ability to accurately represent a variety of functions and operators at different levels of resolution. Moreover, wavelets establish a connection with fast numerical algorithms [5]. Wavelets methods, such as Haar wavelets [6-8], Chebyshev wavelets [9,10], CAS [11,12] and Legendre wavelets [13-15] have been successfully developed to solve the fractional order differential equations and proved to be effective and powerful in simulation of fractional phenomena. A detailed description about wavelets methods in fractional differential equations has been presented in reference [16]. The main attraction of wavelets methods is that it can exploit multi-level parallelism by employing the multi-scale analysis and hierarchy structure of Legendre wavelets.

As an extension of wavelets, multi-wavelets are a discontinuous, orthogonal, compactly supported, multiscale set of functions with vanishing moments and can approximate a large class functions in L2 space by using multiresolution analysis (MRA) in terms of translates of linear combinations of the scaling function vector. Sparse representation of differential and integral operators due to vanishing moments of the wavelets is another property of multi-wavelets [17-23]. What's more, the numerical methods using multi-wavelets as basis are highly stable and the related computation is economic [24]. Legendre multi-wavelets [17] were firstly developed in 1990 by Alpert to provide sparse representations of integral operator for the solution of integral equations and have been successfully used to solve partial differential equations [20-22]. It has been proved that Legendre multi-wavelets can reduce the error to a level below that of wavelets [23-24]. Comparing to Haar wavelets, Legendre multi-wavelets can converge more rapidly and produce piecewise polynomial solutions of any order. What's more, the properties of piecewise polynomials can be used to decrease the saving and computational complexity.

To the best of the authors' knowledge, Legendre multi-wavelets have not been used previously for solving fractional differential equation. Motivated and inspired by the ongoing research, we develop a new spectral collocation method using Legendre multi-wavelets (named LMW) to the solution of fractional differential equation. The remainder of the paper is organized as follows. In section 2, we describe some preliminaries about fractional calculus theory and also illustrate how to construct Legendre multi-wavelets. Fractional integral matrix and nonlinear terms approximation of Legendre multi-wavelets are derived in section 3. The proposed method is detailed in section 4. Five examples are given in Section 5 to demonstrate the validity and applicability of the proposed method. Finally the concluding remarks are given in Section 6.

2 Preliminaries and Notations

2.1 Fractional calculus theory

In this section, some mathematical preliminaries about the fractional calculus theory will be introduction. The readers could see reference [25,26] for more details about fractional calculus theory.

Definition 2.1: A real function $h(x)$, $x>0$, is said to be in the space C_μ, $\mu \in R$, if there exists a real number $p>\mu$, such that $h(x)=x^p h_1(x)$, where $h_1(x)\in C(0,\infty)$, and it is said to be in the space C_μ^n if and only if $h^{(n)}\in C_\mu,\ n\in N$.

Definition 2.2: Riemann-Liouville fractional integral operator (J^α) of order $\alpha \ge 0$, of a function $f\in C_\mu,\ \mu\ge -1$ is defined as,

$$\begin{aligned} &J^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-\tau)^{\alpha-1} f(\tau)d\tau,\ \ x>0,\\ &J^0 f(x)=f(x), \end{aligned} \tag{1}$$

in which $\Gamma(\alpha)$ is the well known Gamma function.

Definition 2.3 The fractional derivative of $f(x)$ in the Caputo sense is defined as [27].

$$\left(D_x^\alpha f\right)(x)=\begin{cases} \dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^x \frac{f^{(m)}(\xi)}{(x-\xi)^{\alpha-m+1}}d\xi, & (\alpha>0,\ \ m-1<\alpha<m) \\ \dfrac{\partial^m f(x)}{\partial x^m}, & \alpha=m \end{cases}, \tag{2}$$

where $f:R\to R,\ \ x\to f(x)$ denotes a continuous (but not necessarily differentiable) function.

Lemma 1. Let $n-1<\alpha\le n,\ \ n\in N,\ \ x>0,\ \ h\in C_\mu^n,\ \ \mu\ge -1$. Then

$$\left(J^\alpha D^\alpha\right)h(x)=h(x)-\sum_{k=0}^{n-1}h^{(k)}\left(0^+\right)\frac{x^k}{k!}. \tag{3}$$

2.2 Legendre multi-wavelets

In this section, Legendre multi-wavelets are introduced and some useful properties are also described. A detailed description about Legendre multi-wavelets can be seen in reference [17-19].

2.2.1 Legendre multi-wavelets

Legendre scaling functions are defined on the interval [0, 1) as follows:

$$\phi_{J,k}^m(x)=\begin{cases} \sqrt{2m+1}\,2^{J/2}L_m\left(2\left(2^J x-k\right)-1\right), & \dfrac{k}{2^J}\le x\le \dfrac{k+1}{2^J} \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

where $m=0,\ 1,\ 2,\cdots,r-1$, $k=0,\ 1,\cdots,2^J-1$ and $J=0,\ 1,\ 2,\cdots$, Here, $L_m(x)$ are the well-known Legendre polynomials of order *m*.

Legendre scaling function vector and Legendre multi-wavelets vector in subspaces $\mathbf{V}_J = \mathbf{V}_{J-1} \oplus \mathbf{W}_{J-1}$ are defined as

$$\mathbf{\Phi}_J(x) = \left[\phi_{J,0}^0(x),\cdots,\phi_{J,0}^{r-1}(x),|,\cdots,\phi_{J,(2^J-1)}^0(x),\cdots,\phi_{J,(2^J-1)}^{r-1}(x)\right] \tag{5}$$

and

$$\begin{aligned}\mathbf{\Psi}_J(x) = [&\phi_{0,0}^0(x),\cdots,\phi_{0,0}^{r-1}(x),|\,\psi_{0,0}^0(x),\cdots,\psi_{0,0}^{r-1}(x)\,|,\\ &\cdots,|\,\psi_{J-1,0}^0(x),\cdots,\psi_{J-1,0}^{r-1}(x)\,|,\cdots,\psi_{J-1,2^{J-1}-1}^0(x),\cdots,\psi_{J-1,2^{J-1}-1}^{r-1}(x)],\end{aligned} \tag{6}$$

respectively.

Legendre scaling function vector $\mathbf{\Phi}_J(t)$ satisfies a two-scale equation of the form

$$\mathbf{\Phi}_J(x) = \sum_{k=0}^{1} \mathbf{H}^{(k)} \mathbf{\Phi}_J(2x-k), \tag{7}$$

where $\mathbf{H}^{(k)}$ is a finite set of matrices,

The two-scale relation of Legendre multi-wavelets vector is given in the following form

$$\mathbf{\Psi}_J(x) = \sum_{k=0}^{1} \mathbf{G}^{(k)} \mathbf{\Phi}_J(2x-k), \tag{8}$$

where $\mathbf{\Psi}_J(x)$ is the vector of corresponding wavelet functions. So the Legendre multi-wavelets can be constructed by Legendre scaling functions by two-scale equation Eq. (8).

A function $f(x)$ in $\mathbf{V}_n^k$ can be approximated by using the Legendre scaling functions as

$$f(x) \approx P_J f(x) = \sum_{k=0}^{2^J-1} \sum_{m=0}^{r-1} c_{J,k}^m \phi_{J,k}^m(x) = \mathbf{C}^{\mathrm{T}} \mathbf{\Phi}_J(x), \tag{9}$$

and the decomposition of $f(x)$ has an equivalent Legendre multi-wavelets given by

$$f(x) \approx P_J f(x) = \sum_{m=0}^{r-1} \left\{ c_{0,0}^m \phi_{0,0}^m(x) + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} d_{j,k}^m \psi_{j,k}^m(x) \right\} = \mathbf{D}^{\mathrm{T}} \mathbf{\Psi}_J(x), \tag{10}$$

where

$$c_{j,k}^m = \left\langle f(x), \phi_{j,k}^m \right\rangle, \quad d_{j,k}^m = \left\langle f(x), \psi_{j,k}^m \right\rangle, \tag{11}$$

in which $\langle .,. \rangle$ denotes the stander inner product of the Hilbert space, and **C** and **D** are $N \times 1$ vectors with $N = r2^J$ given by

$$\mathbf{C}^{\mathrm{T}} = \left[c_{J,0}^{0},\cdots,c_{J,0}^{r-1} \,|\, \cdots \,|\, c_{J,\left(2^{J}-1\right)}^{0},\cdots,c_{J,\left(2^{J}-1\right)}^{r-1} \right], \tag{12}$$

and

$$\mathbf{D}^{\mathrm{T}} = [d_{0,0}^{0},\cdots,d_{0,0}^{r-1},|\, d_{0,0}^{0},\cdots,d_{0,0}^{r-1} \,|,\cdots,|\, d_{J-1,0}^{0},\cdots,d_{J-1,0}^{r-1} \,|,\cdots,d_{J-1,2^{J-1}-1}^{0},\cdots,d_{J-1,2^{J-1}-1}^{r-1}]. \tag{13}$$

The operator P_J is named as the orthogonal projection of function *f* onto space V_J .

For example the cubic Legendre scaling functions consist of four functions are given in the following:

$$\begin{cases} \phi^{0}(x) = 1, & 0 \le x < 1 \\ \phi^{1}(x) = \sqrt{3}(2x-1), & 0 \le x < 1 \\ \phi^{2}(x) = \sqrt{5}\left(6x^{2}-6x+1\right), & 0 \le x < 1 \\ \phi^{3}(x) = \sqrt{7}\left(20x^{3}-30x^{2}+12x-1\right), & 0 \le x < 1 \end{cases}, \tag{14}$$

The corresponding Legendre multi-wavelets $\psi^{0}(x)$, $\psi^{1}(x)$, $\psi^{2}(x)$ and $\psi^{3}(x)$ have the form as

$$\psi^{0}(x) = \begin{cases} -\sqrt{\frac{15}{17}}\left(224x^{3}-216x^{2}+56x-3\right), & 0 \le x < \frac{1}{2} \\ \sqrt{\frac{15}{17}}\left(224x^{3}-456x^{2}+296x-61\right), & \frac{1}{2} \le x < 1 \end{cases}, \tag{15}$$

$$\psi^{1}(x) = \begin{cases} \sqrt{\frac{1}{21}}\left(1680x^{3}-1320x^{2}+270x-11\right), & 0 \le x < \frac{1}{2} \\ \sqrt{\frac{1}{21}}\left(1680x^{3}-3720x^{2}+2670x-619\right), & \frac{1}{2} \le x < 1 \end{cases}, \tag{16}$$

$$\psi^{2}(x) = \begin{cases} -\sqrt{\frac{35}{17}}\left(256x^{3}-174x^{2}+30x-1\right), & 0 \le x < \frac{1}{2} \\ \sqrt{\frac{35}{17}}\left(256x^{3}-594x^{2}+450x-111\right), & \frac{1}{2} \le x < 1 \end{cases}, \tag{17}$$

$$\psi^{3}(x) = \begin{cases} \sqrt{\frac{5}{21}}\left(420x^{3}-246x^{2}+36x-1\right), & 0 \le x < \frac{1}{2} \\ \sqrt{\frac{5}{21}}\left(420x^{3}-1014x^{2}+804x-209\right), & \frac{1}{2} \le x < 1 \end{cases}, \tag{18}$$

which are determined using Eqs. (7) and (8). The matrix $\mathbf{H}^{(0)}$ and $\mathbf{G}^{(0)}$ for $r = 4$ are defined as [24].

$$\mathbf{H}^{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{\sqrt{15}}{4} & \frac{1}{4} & 0 \\ \frac{\sqrt{7}}{8} & \frac{\sqrt{21}}{8} & -\frac{\sqrt{35}}{8} & \frac{1}{8} \end{pmatrix} \text{ and } \mathbf{G}^{(0)} = \begin{pmatrix} 0 & \frac{2}{\sqrt{85}} & 2\sqrt{\frac{3}{17}} & -\sqrt{\frac{21}{85}} \\ -\frac{1}{\sqrt{21}} & -\frac{1}{\sqrt{7}} & -\frac{1}{2}\sqrt{\frac{5}{21}} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{1}{4}\sqrt{\frac{21}{85}} & -\frac{3}{4}\sqrt{\frac{7}{17}} & -\frac{8}{\sqrt{85}} \\ \frac{5}{8}\sqrt{\frac{5}{21}} & \frac{5}{8}\sqrt{\frac{5}{7}} & \frac{23}{8\sqrt{21}} & \frac{\sqrt{15}}{8} \end{pmatrix}. \tag{19}$$

The elements $h_{i,j}^{(1)}$ of matrix $\mathbf{H}^{(1)}$ and $g_{i,j}^{(1)}$ of matrix $\mathbf{G}^{(1)}$ can be calculated by using the following formulae

$$h_{i,j}^{(1)} = (-1)^{i+j} h_{i,j}^{(0)} \tag{20}$$

and

$$g_{i,j}^{(1)} = (-1)^{i+j+K} g_{i,j}^{(0)}, \tag{21}$$

respectively.

<u>2.2.2 Transition matrix of Legendre multi-wavelets</u>

Eq. (7) can be rewritten as

$$\mathbf{\Phi}_k = \mathbf{H}_k \mathbf{\Phi}_{k+1}, \tag{22}$$

where $\mathbf{H}_k$, $k = 1, 2, \cdots, J$ is a $\left(r2^{k-1}, r2^k\right)$ matrix.

Similarly, Eq. (8) can be rewritten as

$$\mathbf{\Psi}_k = \mathbf{G}_k \mathbf{\Phi}_{k+1}, \tag{23}$$

where $\mathbf{G}_k$, $k = 1, 2, \cdots, J$ is a $\left(r2^{k-1}, r2^k\right)$ matrix.

From Eqs. (22) and (23), one can have

$$\mathbf{\Psi}_J = \mathbf{G}\mathbf{\Phi}_{J+1}, \tag{24}$$

where $\mathbf{G}$ is $(N \times N)$ transit matrix and can be calculated as follows [23].

$$\mathbf{G} = \begin{bmatrix} \mathbf{H}_1 \times \mathbf{H}_2 \times \cdots \times \mathbf{H}_J \\ \mathbf{G}_1 \times \mathbf{H}_2 \times \cdots \times \mathbf{H}_J \\ \vdots \\ \mathbf{G}_{J-2} \times \mathbf{H}_{J-1} \times \cdots \times \mathbf{H}_J \\ \mathbf{G}_{J-1} \times \mathbf{H}_J \\ \mathbf{G}_J \end{bmatrix}_{N \times N}. \tag{25}$$

Relations (24) and (25) can be used to develop the fast algorithms for transition between different scales of the multiresolution analysis. It's worth mentioning that the transition operator between coefficients and values is sparse and possess hierarchy structure.

2.3 Block-Pulse functions

The *N*-set of block-pulse functions is defined on $[0,l)$ as follows

$$b_i(x)=\begin{cases}1, & \frac{(i-1)l}{N}\le x<\frac{il}{N},\\ 0, & otherwise,\end{cases} \tag{26}$$

where $i=1,\ 2,\cdots,\ N$. The functions $b_i(x)$ are disjoint and orthogonal. That is, for $x\in[0,l)$ and $i,\ \ j=1,\ 2,\cdots,\ N$.

$$b_i(x)b_j(x)=\begin{cases}b_i(x), i=j\\ 0, \quad i\neq j\end{cases}, \tag{27}$$

and

$$\int_0^l b_i(x)b_j(x)=\begin{cases}\frac{l}{N}, & i=j\\ 0, & i\neq j\end{cases}, \tag{28}$$

The orthogonality property of block-pulse functions is obtained from the disjointness property for $i=1,\ 2,\cdots,\ N$. It is also known that for arbitrary absolutely integrable function $f(x)$ on $[0,l)$ can be expanded in block-pulse functions:

$$f(x)=\boldsymbol{\xi}^{\mathrm{T}}\mathbf{B}_N(x), \tag{29}$$

in which

$$\boldsymbol{\xi}^{\mathrm{T}}=[f_1,f_2,\cdots,f_N] \tag{30}$$

And

$$\mathbf{B}_N(x)=\left[b_1(x),b_2(x),\cdots,b_N(x)\right], \tag{31}$$

where $f_i\left(i=1,\ 2,\cdots N\right)$ are the coefficients of the block-pulse function given by

$$f_i=\frac{N}{l}\int_0^l f(x)b_i(x)dx=\frac{N}{l}\int_{((i-1)/N)l}^{(i/N)l}f(x)b_i(x)dx. \tag{32}$$

So the disjointness property of BPF's follows

$$\mathbf{B}_N(x)\mathbf{B}_N^{\mathrm{T}}(x)\boldsymbol{\xi}=\mathrm{diag}(\boldsymbol{\xi})\mathbf{B}_N(x). \tag{33}$$

3 Nonlinear Term Approximation and Fractional Integral of Legendre Multi-wavelets

3.1 Nonlinear term approximation

The Legendre scale functions can be expanded into *N*-set of block-pulse functions as

$$\mathbf{\Phi}_J(x) = \mathbf{\Phi}_{N\times N}\mathbf{B}_N(x). \tag{34}$$

where $N = 2^J r$.

Taking the collocation points as following

$$x_i = \frac{i - 1/2}{N}, \quad i = 1,\ 2,\ \cdots,\ N. \tag{35}$$

The *N*-square Legendre matrix $\mathbf{\Phi}_{N\times N}$ is defined as:

$$\mathbf{\Phi}_{N\times N} \triangleq \left[\Phi(x_1) \quad \Phi(x_2) \quad \cdots \quad \Phi(x_N)\right] \tag{36}$$

and

$$\mathbf{\Psi}_J(x) = \mathbf{G}\mathbf{\Phi}_{J+1}(x) = \mathbf{G}\mathbf{\Phi}_{N\times N}\mathbf{B}_N(x). \tag{37}$$

The operational matrix of product of Legendre multi-wavelets vector $\mathbf{\Psi}_J(x)$ can be obtained by using the properties of BPFs. Let $f_1(x)$ and $f_2(x)$ are two absolutely integrable functions, which can be expanded in Legendre multi-wavelets as $f_1(x) = \mathbf{F}_1^{\mathrm{T}}\mathbf{\Psi}_J(x)$ and $f_2(x) = \mathbf{F}_2^{\mathrm{T}}\mathbf{\Psi}_J(x)$, respectively.

From Eq. (37), we have

$$\begin{aligned} f_1(x) &= \mathbf{F}_1^{\mathrm{T}}\mathbf{\Psi}_J(x) = \mathbf{F}_1^{\mathrm{T}}\mathbf{G}\mathbf{\Phi}_{N\times N}\mathbf{B}(x) \\ f_2(x) &= \mathbf{F}_2^{\mathrm{T}}\mathbf{\Psi}_J(x) = \mathbf{F}_2^{\mathrm{T}}\mathbf{G}\mathbf{\Phi}_{N\times N}\mathbf{B}(x) \end{aligned}. \tag{38}$$

By employing Lemma 1 in [28] and Eq. (38), we get

$$\begin{aligned} f_1(x)f_2(x) &= \left(\mathbf{F}_1^{\mathrm{T}}\mathbf{G}\mathbf{\Phi}_{N\times N} \otimes \mathbf{F}_2^{\mathrm{T}}\mathbf{G}\mathbf{\Phi}_{N\times N}\right)\mathbf{B}(x) \\ &= \left(\mathbf{F}_1^{\mathrm{T}}\mathbf{G}\mathbf{\Phi}_{N\times N} \otimes \mathbf{F}_2^{\mathrm{T}}\mathbf{G}\mathbf{\Phi}_{N\times N}\right)\mathrm{inv}\left(\mathbf{G}\mathbf{\Phi}_{N\times N}\right)\mathbf{G}\mathbf{\Phi}_{N\times N}\mathbf{B}(x), \\ &= \left(\mathbf{F}_1^{\mathrm{T}}\mathbf{G}\mathbf{\Phi}_{N\times N} \otimes \mathbf{F}_2^{\mathrm{T}}\mathbf{G}\mathbf{\Phi}_{N\times N}\right)\mathrm{inv}\left(\mathbf{G}\mathbf{\Phi}_{N\times N}\right)\mathbf{\Psi}_J(x) \end{aligned} \tag{39}$$

where $\mathbf{A} \otimes \mathbf{B} = \left(a_{ij}b_{ij}\right)_{N\times N}$ is the Hadamara products of matrix $\mathbf{A} = \left(a_{ij}\right)_{N\times N}$ and $\mathbf{B} = \left(b_{ij}\right)_{N\times N}$. Here and after we denote $\mathbf{\Psi}_{N\times N} = \mathbf{G}\mathbf{\Phi}_{N\times N}$.

Fig. 1 shows the sparse structure of matrices **T** and $\mathbf{\Psi}_{N\times N}$. It can be found that the transition matrices **T** and $\mathbf{\Psi}_{N\times N}$ are sparse and possess hierarchy structure which can be used to save memory cost.

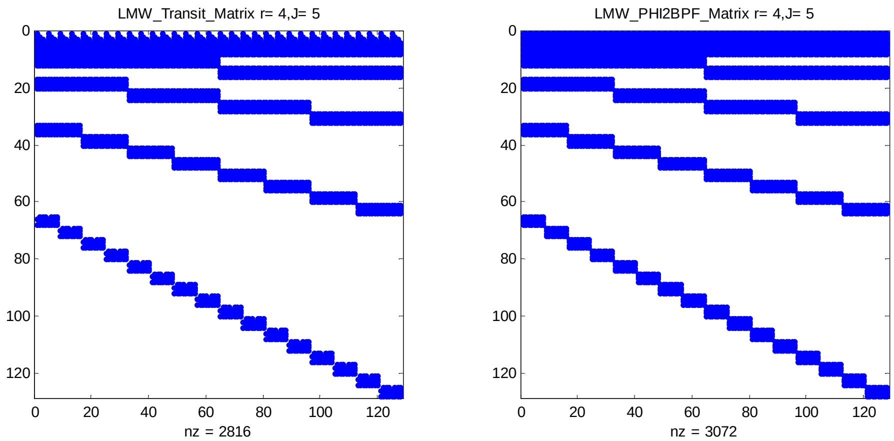

Fig. 1. Sparse structure of matrices of T and $\mathbf{\Psi}_{N\times N}$

3.2 The fractional integral operator of Legendre multi-wavelets

The Legendre multi-wavelets can be expanded into *N*-set of block-pulse functions as

$$\mathbf{\Psi}(x) = \mathbf{\Psi}_{N\times N}\mathbf{B}(x). \tag{40}$$

The fractional integral of block-pulse function vector can be written as

$$(I^{\alpha}\mathbf{B})(x) \simeq \mathbf{P}_B^{\alpha}\mathbf{B}(x), \tag{41}$$

where $\mathbf{P}_B^{\alpha}$ is given in [29] and is defined as

$$\mathbf{P}_B^{\alpha} = \frac{1}{N^{\alpha}}\frac{1}{\Gamma(\alpha+2)}\begin{pmatrix} 1 & \xi_1 & \xi_2 & \cdots & \xi_{N-1} \\ 0 & 1 & \xi_1 & \cdots & \xi_{N-2} \\ 0 & 0 & 1 & \cdots & \xi_{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \tag{42}$$

in which

$$\xi_i = (i+1)^{\alpha+1} - 2i^{\alpha+1} + (i-1)^{\alpha+1}. \tag{43}$$

The fractional integral of vectors $\mathbf{\Psi}_J(x)$ and $\mathbf{\Phi}_J(x)$ can be expressed as

$$(I^{\alpha}\mathbf{\Phi}_J)(x) = \mathbf{P}_{\phi}^{\alpha}\mathbf{\Phi}_J(x), \quad (I^{\alpha}\mathbf{\Psi}_J)(x) = \mathbf{P}_{\psi}^{\alpha}\mathbf{\Psi}_J(x), \tag{44}$$

where $\mathbf{P}_{\phi}^{\alpha}$ and $\mathbf{P}_{\psi}^{\alpha}$ are operational matrices of fractional integral for Legendre scaling functions and Legendre multi-wavelets, respectively. The matrix $\mathbf{P}_{\phi}^{\alpha}$ can be obtained by the following process.

By using Eq. (34)

$$\left(I^{\alpha}\mathbf{\Phi}_J\right)(x) \simeq \left(I^{\alpha}\mathbf{\Phi}_{N\times N}\mathbf{B}\right)(x) = \mathbf{\Phi}_{N\times N}\left(I^{\alpha}\mathbf{B}\right)(x) = \mathbf{\Phi}_{N\times N}\mathbf{P}_B^{\alpha}\mathbf{B}(x) = \mathbf{\Phi}_{N\times N}\mathbf{P}_B^{\alpha}\Phi_{N\times N}^{-1}\mathbf{\Phi}_J(x), \tag{45}$$

So

$$\mathbf{P}_{\phi}^{\alpha} \simeq \mathbf{\Phi}_{N\times N}\mathbf{P}_B^{\alpha}\Phi_{N\times N}^{-1}. \tag{46}$$

From Eqs. (24) and (44), we have

$$\left(I^{\alpha}\mathbf{\Psi}_J\right)(x) = \mathbf{G}\left(I^{\alpha}\mathbf{\Phi}_{J+1}\right)(x) = \mathbf{G}\mathbf{P}_{\phi}^{\alpha}\mathbf{\Phi}_{J+1}(x) = \mathbf{G}\mathbf{P}_{\phi}^{\alpha}\mathbf{G}^{-1}\mathbf{\Phi}_J(x). \tag{47}$$

Finally, we obtain

$$\mathbf{P}_{\psi}^{\alpha} = \mathbf{G}\mathbf{P}_{\phi}^{\alpha}\mathbf{G}^{-1}. \tag{48}$$

4 Numerical Method and Convergence Analysis

4.1 Description of numerical method

Consider the following initial value problem:

$$D^{\alpha}u(x) + N\left[u(x)\right] + L\left[u(x)\right] = g(x), \quad \alpha > 0, \tag{49}$$

$$u^{(k)}(0) = c_k, \quad k = 0, 1, 2, \cdots, m-1, \quad m-1 < \alpha \le m, \tag{50}$$

where L is a linear operator, N is a nonlinear operator, and D^{α} is the Caputo fractional derivative of order α.

Applying the fractional integral operator I^{α} to both sides of (49) and using Lemma 1, yields

$$u(x) + I^{\alpha}\left(N\left[u(x)\right] + L\left[u(x)\right] - R(x)\right) = 0, \quad \alpha > 0, \tag{51}$$

in which $R(x) = \sum_{k=0}^{n-1} u^{(k)}\left(0^{+}\right)\frac{x^k}{k!} + g(x)$.

To solve the problem, $u(x)$ and $R(x)$ are approximated by Legendre multi-wavelets as

$$u(x) = \mathbf{D}^{\mathrm{T}}\mathbf{\Psi}_J(x), \; R(x) = \mathbf{R}^{\mathrm{T}}\mathbf{\Psi}_J(x). \tag{52}$$

Now for the nonlinear part $N\left[u(x)\right]$, by nonlinear term approximation described in Section 3.1, we have

$$N\left[u(x)\right] = \mathbf{N}^{\mathrm{T}}\mathbf{\Psi}_J(x). \tag{53}$$

For the linear part, we have

$$L\left[u(x)\right] = \mathbf{L}^{\mathrm{T}}\mathbf{\Psi}_J(x) = a\mathbf{D}^{\mathrm{T}}\mathbf{\Psi}_J(x). \tag{54}$$

where $\mathbf{L} = a\mathbf{D}$ has a linear relation with **D** and *a* is constant.

Substituting Eq. (52)-(54) into Eq. (49), we can get a system of algebraic equations as

$$\mathbf{D}^{\mathrm{T}} + \left(\mathbf{N}^{\mathrm{T}} + \mathbf{L}^{\mathrm{T}} - \mathbf{R}^{\mathrm{T}}\right)\mathbf{P}_{\psi}^{\alpha} = 0, \quad \alpha > 0, \tag{55}$$

According to the Wu's [30] technology for determining the initial iteration value, the first guess can be determined as

$$u_0(x) = \sum_{k=0}^{n-1} u^{(k)}\left(0^+\right)\frac{t^k}{k!} + J^{\alpha}\left(g_{frac}(x)\right), \tag{56}$$

where $g_{frac}(x)$ is the fractional component of $g(x)$ with respect to *x*. The coefficient matrix $\mathbf{D}^{\mathrm{T}}$ can be computed by using the MATLAB function fsolve( ) or the method described in [31]. The power of LMW depends on the occurrence of the exact solution in Eq. (56) and the accurate representation of fractional integral. If the exact solution exists, LMW can converge very fast to the exact solution. What's more, for the linear fractional equation, the solution can be obtained by following equation,

$$\mathbf{D}^{\mathrm{T}} = \mathbf{R}^{\mathrm{T}}\mathbf{P}_{\psi}^{\alpha}\left(\mathbf{I} + a\mathbf{P}_{\psi}^{\alpha}\right)^{-1}. \tag{57}$$

where **I** is an identical matrix.

4.2 Convergence analysis

Theorem 4.1. Suppose that the function $f(x):[0,1] \to \mathbf{R}$ and $f(x) \in C^{r-1}[0,1]$. Then $P_J f(x)$ approximates f with mean error bounded as follows [19]:

$$\left\| f(x) - P_J f(x) \right\| \le M(r,J), \tag{58}$$

in which

$$M(r,J) = \frac{1}{(r-1)!}\frac{1}{2^{(J+2)(r-1)-1}}\sup_{x\in[0,1]}\left| f^{(r-1)}(x) \right|. \tag{59}$$

Theorem 4.2. Suppose that the function $f(x):[0,1] \to \mathbf{R}$ and $f(x) \in C^{r-1}[0,1]$. Then $P_J f(x)$ converges to f .

Proof. Let $P_J f(x) = y_J^r = \sum_{m=0}^{r-1}\left\{ c_{0,0}^m \phi_{0,0}^m(x) + \sum_{j=0}^{J-1}\sum_{k=0}^{2^j-1} d_{j,k}^m \psi_{j,k}^m(x) \right\}$ be a sequence of partial sums, we have

$$\begin{aligned}\left\langle f(x), y_J^r \right\rangle &= \left\langle f(x), \sum_{m=0}^{r-1}\left\{ c_{0,0}^m \phi_{0,0}^m(x) + \sum_{j=0}^{J-1}\sum_{k=0}^{2^j-1} d_{j,k}^m \psi_{j,k}^m(x) \right\} \right\rangle \\ &= \sum_{m=0}^{r-1}\left\{ \bar{c}_{0,0}^m \left\langle f(x), \phi_{0,0}^m(x) \right\rangle + \sum_{j=0}^{J-1}\sum_{k=0}^{2^j-1} \bar{d}_{j,k}^m \left\langle f(x), \psi_{j,k}^m(x) \right\rangle \right\}. \\ &= \sum_{m=0}^{r-1}\left\{ \left|c_{0,0}^m\right|^2 + \sum_{j=0}^{J-1}\sum_{k=0}^{2^j-1} \left|d_{j,k}^m\right|^2 \right\}\end{aligned} \tag{60}$$

Without loss of generality, assuming that $r_2 > r_1$ and $J_2 > J_1$, then we can get

$$\begin{aligned}\left\| y_{J_1}^{r_1} - y_{J_2}^{r_2} \right\| &= \left\| \sum_{m=0}^{r_1-1}\left\{ c_{0,0}^m \phi_{0,0}^m(x) + \sum_{j=0}^{J_1-1}\sum_{k=0}^{2^j-1} d_{j,k}^m \psi_{j,k}^m(x) \right\} - \sum_{m=0}^{r_2-1}\left\{ c_{0,0}^m \phi_{0,0}^m(x) + \sum_{j=0}^{J_2-1}\sum_{k=0}^{2^j-1} d_{j,k}^m \psi_{j,k}^m(x) \right\} \right\| \\ &= \left\| \sum_{m=r_1}^{r_2-1}\left\{ c_{0,0}^m \phi_{0,0}^m(x) + \sum_{j=J_1}^{J_2-1}\sum_{k=0}^{2^j-1} d_{j,k}^m \psi_{j,k}^m(x) \right\} \right\|^2 \\ &= \left\langle \sum_{m=r_1}^{r_2-1}\left\{ c_{0,0}^m \phi_{0,0}^m(x) + \sum_{j=J_1}^{J_2-1}\sum_{k=0}^{2^j-1} d_{j,k}^m \psi_{j,k}^m(x) \right\}, \sum_{m=r_1}^{r_2-1}\left\{ c_{0,0}^m \phi_{0,0}^m(x) + \sum_{j=J_1}^{J_2-1}\sum_{k=0}^{2^j-1} d_{j,k}^m \psi_{j,k}^m(x) \right\} \right\rangle, \\ &= \sum_{m=r_1}^{r_2-1}\left\{ \left|c_{0,0}^m\right|^2 + \sum_{j=J_1}^{J_2-1}\sum_{k=0}^{2^j-1} \left|d_{j,k}^m\right|^2 \right\}\end{aligned} \tag{61}$$

So, as $N = r2^J \to \infty$, from Bessel's inequality and Theorem 4.1, we have

$$\sum_{m=r_1}^{r_2-1}\left\{ \left|c_{0,0}^m\right|^2 + \sum_{j=J_1}^{J_2-1}\sum_{k=0}^{2^j-1} \left|d_{j,k}^m\right|^2 \right\} = \left\| y_{J_1}^{r_1} - y_{J_2}^{r_2} \right\| \le \left\| y(x) - y_{J_1}^{r_1} \right\| + \left\| y(x) - y_{J_2}^{r_2} \right\| \le M(r_1, J_1) + M(r_2, J_2). \tag{62}$$

It can be found that $\sum_{m=0}^{r-1}\left\{ \left|c_{0,0}^m\right|^2 + \sum_{j=0}^{J-1}\sum_{k=0}^{2^j-1} \left|d_{j,k}^m\right|^2 \right\}$ is convergent, thus { y_J^r } is a Cauchy sequence.

Further

$$\begin{aligned}\left\langle f(x) - \lim_{N\to\infty} y_J^r, \phi_{J,k}^m(x) \right\rangle &= \left\langle f(x), \phi_{J,k}^m(x) \right\rangle - \left\langle \lim_{N\to\infty} y_J^r, \phi_{J,k}^m(x) \right\rangle \\ &= c_{J,k}^m - \left(\lim_{N\to\infty} \sum_{k=0}^{2^J-1}\sum_{m=0}^{r-1} c_{J,k}^m \phi_{J,k}^m(x), \phi_{J,k}^m(x) \right). \\ &= c_{J,k}^m - c_{J,k}^m = 0\end{aligned} \tag{63}$$

It implies that sequence { y_J^r } converge to $f(x)$.

Theorem 4.3. Assume that $f(x) \in C^1[0,1]$, and $\exists M > 0$, $\forall x \in (0,1)$, $\left|f'(x)\right| \le M$, as $N \to \infty$, $\boldsymbol{\xi}^{\mathrm{T}}\mathbf{B}_N(x)$ is point-wise convergent to $f(x)$ with error bounded as follows [28,32].

$$\left\| f(x) - \boldsymbol{\xi}^{\mathrm{T}}\mathbf{B}_N(x) \right\| \le \frac{M}{N}. \tag{64}$$

Theorem 4.1, 4.2 and 4.3 give the guarantee of convergence of SCLWM. It is clear that the approximation will be more accurate if *N* is increased.

5 Applications and Results

In this section, we first give two linear examples with exact solution to analysis the accuracy of LMW with difference values of *r* and *J*, and then use three examples to demonstrate the validity and applicability of LMW in the solution of nonlinear fractional differential equations. It should be note that linear examples are solved by using Eq. (57) while the nonlinear cases using Matlab function fsolve( ).

Table 1. L_2 error of example 1 for different values of *r* and *J*

r	*J* =2	*J* =3	*J* =4	*J* =5	*J* =6	*J* =7	*J* =8
2	0.0041	0.0015	5.2449e-4	1.8684e-4	6.6467e-5	2.3621e-5	8.3868e-6
3	4.0370e-4	1.4915e-4	5.4740e-5	1.9973e-5	7.2508e-6	2.6209e-6	9.4393e-7
4	2.6530e-4	9.8094e-5	3.5978e-5	1.3111e-5	4.7531e-6	1.7158e-6	6.1719e-7

Example 1. Consider the composite fractional oscillation equation [13].

$$D^{0.25}u(x)+u(x)-x^2-\frac{2}{\Gamma(2.75)}x^{1.75}=0\,, \tag{65}$$

with the initial condition $u(0)=0$ and the exact solution is $u(x)=x^2$.

Table 1 shows the L_2 error of example 1 for different values of *r* and *J*. It is easy to deduce that the accuracy of LMW is improving with the increase of *r* and *J* and converge rapidly which depend on the accuracy of the integral of fraction-order. It is worth mentioning that the obtained results agree well with exact solutions even for small values of *r* and *J*.

Example 2. Consider the following fractional linear equation [13,3-35].

$$D^{\alpha}u(x)+u(x)=0,\ \ 0<x<1,\ \ 0<\alpha\le 2\,, \tag{66}$$

$$u(0)=1,\ \ u'(0)=0, \tag{67}$$

The exact solution of this problem is $u(x)=\sum_{k=0}^{\infty}\frac{\left(-x^{\alpha}\right)^k}{\Gamma(\alpha k+1)}$.

Table 2. L_2 errors for example 2

r	α	*J* =4	*J* =5	*J* =6	*J* =7	*J* =8
3	1	4.8361e-05	1.7182e-05	6.0904e-06	2.1561e-06	7.6280e-07
	2	2.0809e-05	7.3880e-06	2.6177e-06	9.2652e-07	3.2776e-07
4	1	3.1739e-05	1.1219e-05	3.9663e-06	1.4023e-06	4.9578e-07
	2	1.3633e-05	4.8198e-06	1.7040e-06	6.0246e-07	2.1300e-07

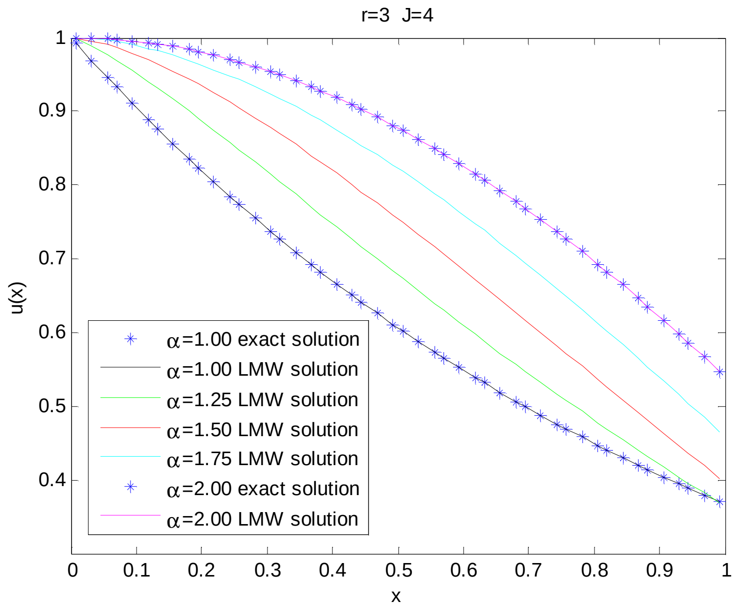

Fig. 2. Numerical solutions of example 2 for difference α

Table 3. Numerical results of Example 2 for α =0.50 and 0.75 with *r*=3 and *J*=5

α		0.50			0.75	
x	**JOM [33]**	**SFJTM [34]**	**LMW**	**JOM [33]**	**SFJTM [34]**	**LMW**
0.1	0.737355	0.723578	0.723585	0.830145	0.828251	0.828258
0.2	0.633468	0.643788	0.643789	0.731281	0.732585	0.732588
0.3	0.593029	0.592018	0.592019	0.660086	0.660337	0.660339
0.4	0.563169	0.553606	0.553606	0.603789	0.602121	0.602122
0.5	0.512164	0.523157	0.523156	0.552192	0.553603	0.553603
0.6	0.499543	0.498025	0.498024	0.512051	0.512285	0.512285
0.7	0.485038	0.476703	0.476703	0.477957	0.476555	0.476555
0.8	0.447227	0.458246	0.458246	0.44395	0.445292	0.445293
0.9	0.452686	0.442021	0.442021	0.418733	0.417682	0.417682
1.0	0.439646	0.427584	0.427583	0.393598	0.393108	0.393108

We solved the problem by applying the method described in Section 4. Fig. 2 exhibits the numerical results for different values of α with *r*=3 and *J*=4. The L_2 errors for α =1.0 and α =2.0 are shown in Table 2. From Fig. 2, one can see that LMW can achieve a good approximation with the exact solution for α =1.0 and α =2.0. In Table 3, we compare our numerical results with those obtained by the shifted fractional-order Jacobi orthogonal functions (SFJTM) [34] and by the Jacobi operational matrix method (JOM) [33]. From Table 3, it is clear that LMW can achieve a good approximation of solution nearly the same as SFJTM and better than JOM.

Table 4. L_2 errors of Example 3 for different values of *r* and *J*

r	***J*** **=1**	***J*** **=2**	***J*** **=3**	***J*** **=4**	***J*** **=5**	***J*** **=6**	***J*** **=7**
2	0.0134	0.0048	0.0017	6.2198e-4	2.2195e-4	7.9073e-05	2.8133e-05
3	0.0014	5.4055e-4	2.0460e-4	7.5963e-5	2.7894e-5	1.0168e-05	5.5354e-15
4	9.7490e-4	3.6690e-4	1.3642e-4	5.0248e-5	1.8373e-5	4.4024e-15	6.3071e-15

Example 3. Consider the composite fractional oscillation equation [14].

$$D^{0.25}u(x)+u^2(x)=\frac{2}{\Gamma(2.75)}x^{1.75}+x^4, \tag{68}$$

with the initial condition $u(0)=0$ and the exact solution is $u(x)=x^2$.

Table 4 show the L_2 errors of example 3 for different values of r and J. It is also support the conclusion that the accuracy of LMW is improving with the increase of r and J and converge rapidly. It can be found that LMW can converge to the exact solution with large values of r and J. This is because that the exact solution exists in first guess defined in Eq. (56) and the error of fractional integral operator is smaller than the tolerance in Matlab function fsolve( ).

Example 4. Consider the following nonlinear fractional Riccati equation [10,36,37]:

$$D^{\alpha}u(x)+u^2(x)=1,\ 0<\alpha\le 1,\ 0<x<1, \tag{69}$$

subject to the initial state $u(0)=0$ .The exact solution of this problem, when α=1, is $u(x)=(e^{2x}-1)/(e^{2x}+1)$.

Fig. 3 exhibits the approximate solutions of Example 4 for difference α with r=3 and J=6. It can be found that LMW's solution is in very good agreement with the exact solution for α=1.00. Therefore, numerical results for α=0.25, 0.50 and 0.75 are also credible. Table 5 compares LMW with variational iteration method using fractional-order Legendre Functions [36]. It can be found that our result is as accurate as those in Ref [36] in the case of α=1.00 and nearly the same for α=0.25, 0.50 and 0.75. This demonstrates the importance of presented numerical scheme in solving nonlinear fractional differential equations.

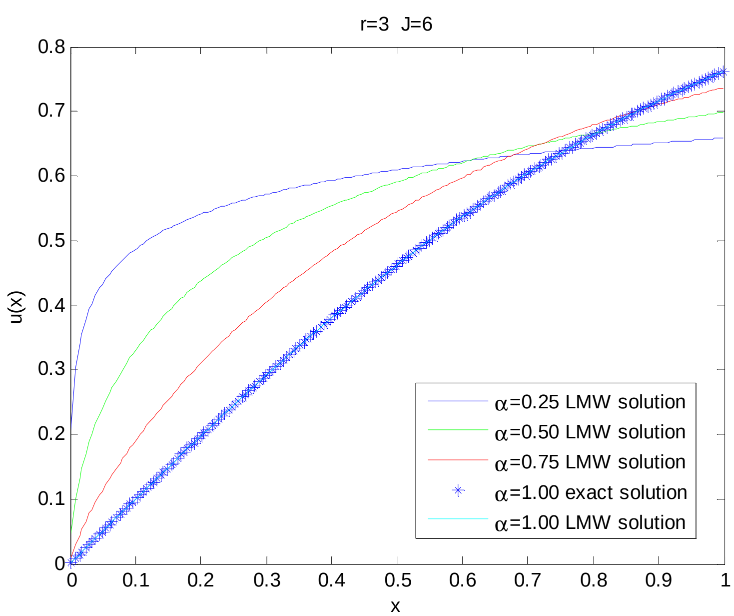

Fig. 3. Numerical solutions of example 4 for difference α

Table 5. Numerical results of Example 4 for *r*=3 and *J*=6 with comparison to Ref. [36]

x	α =0.50		α =0.75		α =1.00		
	Ref. [36]	**LMW**	**Ref. [36]**	**LMW**	**Ref. [36]**	**LMW**	**Exact**
0.1	0.330108	0.330107	0.190101	0.190100	0.099668	0.099668	0.099668
0.2	0.436839	0.436839	0.309976	0.309975	0.197375	0.197375	0.197375
0.3	0.504889	0.504890	0.404615	0.404614	0.291313	0.291312	0.291313
0.4	0.553782	0.553782	0.481632	0.481631	0.379949	0.379949	0.379949
0.5	0.591195	0.591194	0.545090	0.545089	0.462117	0.462117	0.462117
0.6	0.621014	0.621014	0.597783	0.597783	0.537050	0.537049	0.53705
0.7	0.645485	0.645486	0.641820	0.641820	0.604368	0.604367	0.604368
0.8	0.666020	0.666019	0.678850	0.678849	0.664037	0.664037	0.664037
0.9	0.683554	0.683552	0.710175	0.710175	0.716298	0.716298	0.716298

Example 5. Consider fractional Riccati equation [36,37].

$$D^{\alpha}u(x) - 2u(x) + u^2(x) = 1,\ \ 0 < \alpha \le 1, \tag{70}$$

subject to the initial condition $u(0)=0$, for $\alpha = 1$ the exact solution is $u(x) = 1 + \sqrt{2}\tanh\left[\sqrt{2}x + \frac{1}{2}\log\left(\left(\sqrt{2}-1\right)/\left(\sqrt{2}+1\right)\right)\right]$.

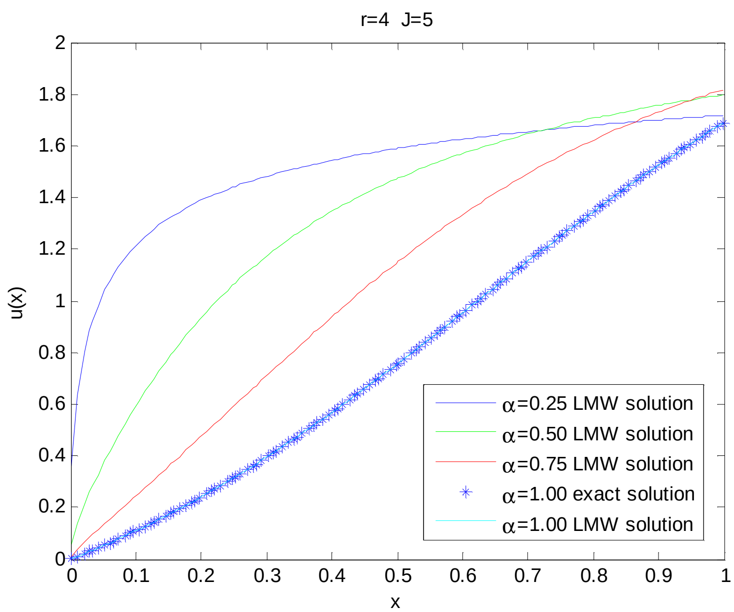

Fig. 4. Numerical solutions of example 5 for difference *α*

Fig. 4 gives approximation solution of fractional Riccati differential equation for Example 5. From Fig. 4 it can found that the numerical solution is in very good agreement with the exact solution when $\alpha = 1.0$. Therefore, it should be hold that the solution for $\alpha = 0.25$, 0.50 and 0.75 is also credible. Table 6 compares the results of LMW with those obtained by variational iteration method using fractional-order Legendre Functions. It can be found that our result is more accurate than that in Ref [36] for $\alpha = 1.0$. So it can hold that LMW's results are better for $\alpha = 0.25$, 0.50 and 0.75. This demonstrates the importance of presented numerical scheme in solving nonlinear fractional differential equations.

Table 6. Numerical results of Example 5 for *r*=4 and *J*=5 with comparison to Ref. [36]

x	***α*=0.50**		***α*=0.75**		***α*=1.0**		
	Ref[36]	**LMW**	**Ref[36]**	**LMW**	**Ref[36]**	**LMW**	**exact**
0.1	0.592833	0.592785	0.245446	0.245443	0.110308	0.110300	0.110295
0.2	0.933104	0.933200	0.475051	0.475108	0.241990	0.241983	0.241976
0.3	1.174069	1.173996	0.710050	0.710032	0.395119	0.395111	0.395104
0.4	1.346694	1.346661	0.938523	0.938543	0.567830	0.567819	0.567812
0.5	1.473790	1.473890	1.149016	1.149066	0.756032	0.756020	0.756014
0.6	1.570577	1.570573	1.334339	1.334336	0.953583	0.953571	0.953566
0.7	1.646302	1.646199	1.491949	1.491927	1.152968	1.152953	1.152946
0.8	1.706644	1.706880	1.622950	1.622992	1.346381	1.346367	1.346363
0.9	1.756349	1.756643	1.730575	1.730612	1.526927	1.526914	1.526911

6 Conclusion

In this work Legendre multi-wavelets operational matrix of fractional order integration is derived and used to solve fractional differential equations. By using the two-scales difference relations of Legendre multi-wavelets and properties of BPFs, the operational matrices of fractional integration and nonlinear term expansion coefficients vector are derived. Consequently, the solution of fractional differential equation is converted to the solution of a sparse nonlinear system of algebraic equations. The achieved results are compared with exact solutions and with the solutions obtained by some other numerical methods. The numerical results show that LMW is accuracy and applicable even for small values of r and J. what's more, LMW can get the exact solution when exact solution exists in first guess defined in Eq. (56) and the error of fractional integral operator is smaller than the tolerance in Matlab function f solve( ). It is worth mentioning that the proposed algorithms can be extended to solve a number of fractional partial differential equations.

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Competing Interests

Author has declared that no competing interests exist.

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