Journal of Scientific Research & Reports
13(6): 1-15, 2017; Article no.JSRR.32122
ISSN: 2320-0227

SCIENCEDOMAIN *international*
www.sciencedomain.org

Harmonic and Sub-Harmonic Periodic Solutions $(\frac{1}{2}, \frac{1}{3})$ of Generalized Mathieu-Van der Pol-Duffing Equations

A. M. El-Naggar[1], K. M. Khalil[1] and A. M. Omran[1*]

[1] *Department of Mathematics, Faculty of Science, Benha University, Benha 13518, Egypt.*

Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/JSRR/2017/32122
Editor(s):
(1) Narayan Thapa, Department of Mathematics and Computer Science, Minot State University, 58707 Minot, ND, US.
Reviewers:
(1) Md. Alal Hosen, Rajshahi University of Engineering and Technology, Bangladesh.
(2) Nityanand P. Pai, Manipal University, Manipal, India.
(3) Rashmi Awad, Devi Ahilya University, Indore, India.
Complete Peer review History: http://www.sciencedomain.org/review-history/18720

Received: 9th February 2017
Accepted: 4th April 2017
Published: 21st April 2017

Original Research Article

ABSTRACT

The frequency-locking area of harmonic and subharmonic $(\frac{1}{2}, \frac{1}{3})$ solutions in a fast harmonic excitation Mathieu-Van der PolDuffing equation is studied. A perturbation technique is then performed on the slow dynamic near the harmonic and subharmonic $(\frac{1}{2}, \frac{1}{3})$ solutions, to obtain reduced slow flow equations governing the modulation of amplitude and phase of the corresponding slow dynamics. Results show that fast harmonic excitation can change the nonlinear characteristic spring behavior from softening to hardening and causes the entrainment regions to shift. Numerical solutions are represented the analytical results.

Keywords: MEMS; multiple scales method; fast excitation; slow motion and parametric forcing.

**Corresponding author: E-mail: amiramasoud.am@gmail.com;*

1 INTRODUCTION

In this paper, we study the effect of a fast harmonic excitation(FHE) on the entrainment area of the harmonic and subharmonic solutions of order ($\frac{1}{2}, \frac{1}{3}$) to a Mathieu-Van Der Pol-Duffing solution. Entrainment or frequency-locking phenomenon was investigated by many authors. Advances in micro fabrication technology have enabled the design and fabrication of Micro-Electro- Mechanical Systems(MEMS) devices in favor of multitude engineering fields that include telecommunications, radar systems, and personal mobiles. While providing improvements, more advanced solutions were desired for broadband performance, and the exploitation of nonlinearity became a subsequent focus. To date, a number of nonlinear energy harvesting studies have been conducted, mostly focusing on the mono stable Duffing [1, 2]. Sebald et al. [3] described a similar technique whereby an impulsive voltage could be applied in the harvesting circuit to achieve the same objective. Among the numerous actuation methods for MEMS devices is electrostatic actuation, which is the most well established technique because of its simplicity and high efficiency [4]. Their design is appropriate for highly tunable 4 microbeams, which offer minimal packaging constraints, low-power consumption, low damping, ease of parameter tuning, and relatively simple integration with electronics. Rhoads and co-workers [5, 6] described a filter design based on the nonlinear response of parametrically-excited MEMS oscillators that have significant potential in many communications applications. Alsaleem et al. [7] investigated the nonlinear phenomena, including primary resonance, superharmonic and subharmonic resonances, in electrostatically actuated resonators both experimentally and theoretically. Harmonic, subharmonic and super-harmonic resonance of weakly nonlinear dynamical system subjected to external excitation, parametric excitation or both are investigated by Elnaggar et al. [8]. Zhang and Tang [9] investigated the chaotic dynamics and global bifurcations of the suspended inclined cable under combined parametric and external excitations.A theoretical discussion and some numerical results relating to a nonlinear state designed for shallow cable vibration are presented and studied by Faravelli and Ubertini [10]. Pandey et al. [11] studied the entrainment in a MEMS to a limit cycle by using the perturbation analysis, also [12] studied frequency-locking in a forced Mathieu van der Pol Duffing system near the principal resonances and provided application in optically driven MEMS resonators. Belhaq [13, 14] investigated 2:2:1 solutions in quasi periodic Mathieu equation. Mohamed et al. [15] 2:1 and 1:1 solutions in Mathieu-Van-der-Pol-Duffing oscillator.

2 PERTURBATION ANALYSIS

In this paper, we study the mathematical model is represented the dynamical behavior of this micro dynamical system is the following weakly nonlinear second order differential equation

$$\begin{aligned} &x'' + \omega_o^2 x - \epsilon(\alpha - \xi x^2)x' - \epsilon h x \cos(\Omega t) + \\ &\epsilon(\alpha_1 x^2 - \alpha_2 x^3 + \alpha_3 x^4) - \epsilon F_1 \cos(\Omega t) \\ &- \epsilon F_2(-\frac{1}{2}x^2 + \frac{1}{24}x^4)\cos(\Omega t) = 0 \end{aligned} \tag{2.1}$$

Eq.(2.1) is a generalized Mathieu-Van der PolDuffing equation subjected a weakly nonlinear parametric excitation and external excitation [15], ϵ is a small parameter $\epsilon \ll 1$, ω_o is the linear natural frequency, Ω is frequency of the external excitation, where damping α, ξ, α_1, α_2 and α_3 are the coefficients of the nonlinear terms, excitation amplitudes. h, F_1 and F_2 are constants represents the coefficient, of linear parametric and nonlinear parametric excitation and external excitation, respectively. Let

$$x(t;\epsilon) = x_o(T_o, T_1) + \epsilon x_1(T_o, T_1) + O(\epsilon^2), T_n = \epsilon^n t, \tag{2.2}$$

where $T_o = t$ is the first scale associated with changes occurring at the frequencies ω_o and Ω , and $T_1 = \epsilon t$ is a slow scale associated with modulations in the amplitude.

$$\frac{d}{dt} = D_o + \epsilon D_1 + ..., \frac{d^2 dt^2}{=} D_o^2 + 2\epsilon D_o D_1 + ..., \tag{2.3}$$

where $D_n = \frac{\partial}{\partial T_n}$. Substituting Eqs.(2.2) and (2.3) into Eq.(2.1) and equating terms with the same power of on both sides, we obtain a system of linear partial differential equations

$$O(1) : D_o^2 x_o + \omega_o^2 x_o = 0 \tag{2.4}$$

$$O(\epsilon): D_o^2 x_1 + \omega_o^2 x_1 = -[2D_0 D_1 x_o + (-\alpha + \xi x_o^2) D_1 x_o + \alpha_1 x_o^2 + (-F_1 - h x_o + \frac{1}{2} F_2 x_o^2 - \frac{1}{24} F_2 x_o^4) \cos[\Omega T_o]] - \alpha_2 x_o^3 + \alpha_3 x_o^4 \tag{2.5}$$

The solution of Eq.(2.4) can be expression the form

$$x_o(T_o, T_1) = A(T_1) e^{i\omega_o T_o} + \bar{A}(T_1) e^{-i\omega_o T_o}, \tag{2.6}$$

where A is the amplitude of the response and is a function of T_1 and $\bar{A}$ is the complex conjugate of A, substitute Eq.(2.6) into Eq.(2.5), we get

$$\begin{aligned} D_o^2 x_1 + \omega_o^2 x_1 = & -e^{i\omega_o T_o}(-i\alpha A\omega_o - 3A^2\alpha_2\bar{A} + i\xi A^2\omega_o\bar{A} + 2i\omega_o A') \\ & - 2A\alpha_1\bar{A} - (-\frac{F_1}{2} + \frac{1}{2} A F_2 \bar{A} - \frac{1}{8} A^2 F_2 \bar{A}^2) e^{i\Omega T_o} \\ & + \frac{1}{2} h\bar{A} e^{i(\Omega - \omega_o)T_o} - 6A^2\alpha_3\bar{A}^2 + \frac{1}{48} F_2 \bar{A}^4 e^{i(\Omega - 4\omega_o)T_o} \\ & + (\frac{1}{4} F_2 \bar{A}^2 - \frac{1}{12} A F_2 \bar{A}^3) e^{i(\Omega - 2\omega_o)T_o} + NST. + c.c, \end{aligned} \tag{2.7}$$

where NST denotes the terms does not produce secular terms and $c.c$ denotes the complex conjugate. Any particular solution of Eq.(2.7) contains secular terms, which are generated by the first term on the right-hand side of Eq.(2.7). Moreover, it may contains small-divisor terms depending on the solution condition. Eq.(2.7) contain harmonic solution and two cases subharmonic solutions $(\Omega \approx n\omega_o)$; $n = 1, 2, 3$. In this paper we restricted our attention to harmonic and two cases of subharmonic solutions of order $(\frac{1}{2}, \frac{1}{3})$.

3 HARMONIC SOLUTION $(\Omega \approx \omega_o)$

In this section, we study harmonic solution of Eq.(2.1) i.e periodic solution with period equal to the period of the excitation term $(\Omega \approx \omega_o)$. Introducing the detuning parameter σ to convert the small divisor term into secular term in Eq.(2.7) i.e

$$\Omega = \omega_o + \epsilon\sigma \tag{3.1}$$

Eliminating the secular terms, we get

$$i\alpha A\omega_o - 2i\omega_o A' + 3A^2\alpha_2\bar{A} - i\xi A^2\omega_o\bar{A} + \lambda e^{i\sigma T_1} = 0, \tag{3.2}$$

where $\lambda = \frac{1}{2}(F_1 - (\bar{A} - \frac{1}{4}A^2\bar{A}^2)F_2)$. Using the polar form $A(T_1) = \frac{1}{2}a(T_1)e^{i\beta(T_1)}$, we obtain

$$\acute{a} = -\frac{1}{8}a^3\xi + \frac{1}{2}a\alpha + \frac{1}{\omega_o}(\frac{F_1}{2} - \frac{a^2F_2}{8} + \frac{a^4F_2}{128})\sin\gamma, \tag{3.3}$$

$$a\gamma' = \frac{3a^3\alpha_2}{8\omega_o} + a\sigma + \frac{1}{\omega_o}(\frac{F_1}{2} - \frac{a^2F_2}{8} + \frac{a^4F_2}{128})\cos\gamma, \tag{3.4}$$

where $\gamma = \sigma T_1 - \beta$ for steady state solution, $\acute{a} = \acute{\gamma} = 0$, in Eqs.(3.3) and (3.4), we obtain

$$\frac{1}{\omega_o}(\frac{F_1}{2} - \frac{a^2F_2}{8} + \frac{a^4F_2}{128})\sin\gamma = \frac{1}{8}a^3\xi - \frac{1}{2}a\alpha \tag{3.5}$$

$$\frac{1}{\omega_o}(\frac{F_1}{2} - \frac{a^2F_2}{8} + \frac{a^4F_2}{128})\cos\gamma = -a\sigma - \frac{3a^3\alpha_2}{8\omega_o} \tag{3.6}$$

Eqs.(3.5) and (3.6) show that there are no trivial solution at $a = 0$. For non-trivial solution i.e. at $a \neq 0$, eliminating γ from Eq.(3.5) and Eq.(3.6), we get the following the frequency-response equation

$$(\frac{1}{8}a^3\xi - \frac{1}{2}a\alpha)^2 + (a\sigma + \frac{3a^3\alpha_2}{8\omega_o})^2 = (\frac{1}{\omega_o}(\frac{F_1}{2} - \frac{a^2F_2}{8} + \frac{a^4F_2}{128}))^2 \tag{3.7}$$

i.e.

$$\sigma = \frac{-48a^4\alpha_2\omega_o \pm \sqrt{z_4F_1^2 + z_1F_1F_2 + z_2F_2^2 + z_3}}{128a^2\omega_o^2}, \tag{3.8}$$

where $z_1 = (-2048a^4 + 128a^6)\omega_o^2, z_2 = (256a^6 - 32a^8 + a^10)\omega_o^2$,
$z_3 = (-256a^8\zeta^2 + 2048a^6\zeta\alpha - 4096a^4\alpha^2)\omega_o^4$ and $z_4 = 4096a^2\omega_o^2$.

A first-order approximation for the solution of Eq.(2.1) can be derived as

$$x = a\cos(\Omega t - \gamma) + O(\epsilon) \tag{3.9}$$

To determined the stability of the nontrivial solution, let

$$a = a_o + a_1(T_1) \qquad \& \qquad \gamma = \gamma_o + \gamma_1(T_1), \tag{3.10}$$

where a_o and γ_o are given by Eqs.(3.5) and (3.6). Inserting Eq.(3.10) into Eq.(3.3) and Eq.(3.4) and using Eqs.(3.5) and (3.6) and keeping only the linear terms in a_1 and γ_1, we get

$$\acute{a_1} = \frac{(\rho_1F_1 + \rho_1F_2)}{\rho_8}a_1 + \frac{(\rho_3F_1 + \rho_4F_4)}{\rho_8}\gamma_1 \tag{3.11}$$

$$\acute{\gamma_1} = \frac{(\rho_5F_1 + \rho_6F_2)}{a_o\rho_8}a_1 + (\rho_7)\gamma_1 \tag{3.12}$$

where $\rho_1 = -192a_o^2\xi\omega_o + 256\alpha\omega_o$, $\quad \rho_2 = 16a_o^4\xi\omega_o + a_o^6\xi\omega_o + 64a_o^2\alpha\omega_o - 12a_o^4\alpha\omega_o$,
$\rho_3 = -192a_o^3\alpha_2 - 512a_o\sigma\omega_o$,
$\rho_4 = 48a_o^5\alpha_2 - 3a_o^7\alpha_2 + 128a_o^3\sigma\omega_o - 8a_o^5\sigma\omega_o$,
$\rho_5 = 64(9a_o^2\alpha_2 + 8\sigma\omega_o)$, $\quad \rho_6 = -a_o^2(3a_o^2(16 + a_o^2)\alpha_2 + 8(-16 + 3a_o^2)\sigma\omega_o)$,
$\rho_7 = -\frac{a_o^2\xi - 4\alpha}{8}$ and $\rho_8 = 8(64F_1 - 16a_o^2F_2 + a_o^4F_2)\omega_o$

Eqs.(3.11) and (3.12) admit solution of the form $(a_1, \phi_1) \propto (c_1, c_2)e^{\theta T_1}$ where (c_1, c_2) are constants. Provided that,

$$\theta = z_{17} \pm \frac{1}{8}\sqrt{\frac{1}{z_5^2\omega_o^2}(z_{18} + z_{19} + z_{20} + +z_{13}F_1F_2\omega_o^2)}, \tag{3.13}$$

where

$z_5 = 64F_1 - 16a^2F_2 + a^4F_2, z_6 = -110592a^4, z_7 = 36864a^6 - 1152a^8$,
$z_8 = -2304a^8 + 9a^{12}, z_9 = -393216a^2, z_{10} = 98304a^4$,
$z_{11} = -1536a^8 + 96a^{10}, z_{12} = 4096a^4\zeta^2 - 262144\sigma^2$,
$z_{13} = -128a^8b^2 - 8192a^4\xi\alpha + 1024a^6\xi\alpha + 8192a^4\sigma^2$,
$z_{14} = a^{12}\xi + 128a^8\alpha - 16a^{10}\alpha, z_{15} = 4096a^4 - 1024a^6 + 64a^8$,
$z_{16} = 16384a^4 - 4096a^6 + 192a^8, z_{17} = \frac{-32a^2\xi F_1 + 64\alpha F_1 + 4a^4\xi F_2 - a^4\alpha F_2}{2z_5}$,
$z_{18} = z_6F_1^2\alpha_2^2 + z_{12}\omega_o^2, z_{19} = (z_7F_1 + z_8F_2)F_2\alpha_2^2, z_{20} = (z_9F_1 + z_{10}F_2)\sigma F_1\alpha_2\omega_o$

and

$z_{21} = z_{11}\sigma F_2^2\alpha_2\omega_o + z_{14}\xi F_2^2\omega_o^2 + z_{15}\alpha^2F_2^2\omega_0^2 + z_{16}\sigma^2F_2^2\omega_o^2$.

Consequently, a solution is stable if and only if the real parts of eigenvalues Eq.(3.13) are less than or equal to zero.

4 SUBHARMONIC SOLUTION OF ORDER $(\frac{1}{2})$ $(\Omega \approx 2\omega_o)$

In this section, we study subharmonic solution of order $\frac{1}{2}$. i.e periodic solutions with period equal two multiple of the period of the excitation term i.e ($\Omega \approx 2\omega_o$). Introducing the detuning parameter σ_1 to convert the small divisor term into secular term in Eq.(2.7).
i.e

$$\Omega = 2\omega_o + \epsilon\sigma_1, \tag{4.1}$$

and, we can write

$$(\Omega - \omega_o)T_o = \omega_o T_o + \epsilon\sigma_1 T_o = \omega_o T_o + \sigma_1 T_1. \tag{4.2}$$

Using Eq.(4.2), the small-divisor term arising from $e^{i(\Omega-2\omega_o)}$ in Eq.(2.7) can be transformed into a secular term. Then, eliminating the secular terms yield

$$-2i\omega_o A' + i\alpha A\omega_o + 3A^2\alpha_2\bar{A} - i\xi A^2\omega_o\bar{A} + \frac{1}{2}h\bar{A}e^{i\sigma_1 T_1} = 0. \tag{4.3}$$

By using

$$A(T_1) = \frac{1}{2}a(T_1)e^{i\beta(T_1)}, \tag{4.4}$$

where a and β are real. Inserting Eq.(4.4) into Eq.(4.3) are real and imaginary parts, we obtain

$$\acute{a} = -\frac{1}{8}a^3\xi + \frac{1}{2}a\alpha + \frac{1}{4\omega_o}ah\sin\phi. \tag{4.5}$$

$$a\acute{\phi} = \frac{3a^3\alpha_2}{4\omega_o} + a\sigma_1 + \frac{1}{2\omega_o}ah\cos\phi, \tag{4.6}$$

where a and ϕ are the amplitude and the phase,

$$\phi = \sigma_1 T_1 - 2\beta. \tag{4.7}$$

It is obvious that, Eqs.(4.5) and (4.6) have a trivial solution which of corresponds to the trivial steady state solution. Non-trivial steady state solution correspond to the non-trivial fixed points (Equilibrium points) of Eqs.(4.5) and (4.6). That is, they satisfy $\acute{a} = \acute{\phi} = 0$, and are given by

$$\frac{1}{4\omega_o}a_o h\sin\phi_o = \frac{1}{8}a_o^3\xi - \frac{1}{2}a_o\alpha. \tag{4.8}$$

$$\frac{1}{4\omega_o}a_o h\cos\phi_o = -\frac{3a_o^3\alpha_2}{8\omega_o} - \frac{1}{2}a_o\sigma_1. \tag{4.9}$$

Eliminating $\sin\phi_o$ and $\cos\phi_o$ from Eqs.(4.8) and (4.9) yields the frequency-response equation

$$(\frac{1}{8}a_o^3\xi - \frac{1}{2}a_o\alpha)^2 + (-\frac{3a_o^3\alpha_2}{8\omega_o} - \frac{1}{2}a_o\sigma_1)^2 = (\frac{1}{4\omega_o}a_o h)^2 \tag{4.10}$$

i.e.

$$\sigma_1 = \frac{-3a^2\alpha_2\omega_o \pm \sqrt{4h^2\omega_o^2 - a^4\xi^2\omega_o^4 + 8a^2\xi\alpha\omega_o^4 - 16\alpha^2\omega_o^4}}{4\omega_o^2} \tag{4.11}$$

Then, the first-order uniform expansion of the solution (first approximation) of Eq.(2.1) is given by

$$x = a\cos(\frac{1}{2}\Omega t - \frac{1}{2}\phi) + O(\epsilon) \tag{4.12}$$

The analysis of the stability of the trivial solutions is equivalent to the analysis of the linear solutions of Eq.(4.3) by neglecting the nonlinear terms we get

$$-2i\omega_o A' + i\alpha A\omega_o + \frac{1}{2}h\bar{A}e^{i\sigma_1 T_1} = 0 \tag{4.13}$$

To determine the stability of the trivial steady state solution, it is convenient to rewrite A in the form

$$A = (B(T_1) + ib(T_1))e^{\frac{1}{2}i\sigma_1(T_1)}, \tag{4.14}$$

where B and b are real and imaginary parts and get

$$\acute{b} + \frac{\alpha}{2}b - \Gamma_1 B = 0, \tag{4.15}$$

$$\acute{B} - \frac{\alpha}{2}B - \Gamma_2 b = 0, \tag{4.16}$$

where $\Gamma_1 = (\frac{\omega_o\sigma_1 + \frac{1}{2}h}{2\omega_o})$ and $\Gamma_2 = (\frac{\omega_o\sigma_1 - \frac{1}{2}h}{2\omega_o})$. Eqs.(4.15)and(4.16) admit solution of the form $(B, b) \propto (B, b)e^{\theta_o T_1}$, where (B, b) are constant. The eigenvalues of the coefficient matrix of Eqs.(4.15) and (4.16) are

$$\theta_o = \pm\sqrt{\frac{\alpha^2}{4} + \Gamma_1\Gamma_2}. \tag{4.17}$$

Then, the trivial solution is stable if the real parts of both eigenvalues are less than or equal zero. To determine the stability of the non-trivial steady state solutions given by Eqs.(4.8) and (4.9). Let

$$a = a_o + a_1(T_1) \qquad \& \qquad \phi = \phi_o + \phi_1(T_1), \tag{4.18}$$

where a_o and ϕ_o correspond to a nontrivial steady state solutions and a_1 and ϕ_1 are perturbations which are assumed to be small compared with a_o and ϕ_o. Substituting Eq.(4.18) into Eqs.(4.5) and (4.6) and linearizing the resulting equations, we obtain

$$\acute{a_1} = -(\frac{a_o^2\xi}{4})a_1 - (\frac{3a_o^3\alpha_2 + 2a_o\sigma_1\omega_o}{8\omega_o})\phi_1, \tag{4.19}$$

$$\acute{\phi_1} = -(\frac{3a_o\alpha_2}{2\omega_o})a_1 - (\frac{a_o^2\xi - 4\alpha}{4})\phi_1. \tag{4.20}$$

Substituting $a_1 = \Gamma_1 e^{\theta T_1}$ and $\gamma_{11} = \Gamma_2 e^{\theta T_1}$ into Eq.(4.19) and Eq.(4.20). We get

$$\begin{aligned} \Gamma_1(-6a_0\alpha_2) + \Gamma_2(\left(a_0^2 b - 4\alpha + 4\theta\right)\omega_0) &= 0, \\ \Gamma_1\left(2a_0^2 b\omega_0 + 8\theta\omega_0\right) + \Gamma_2\left(3a_0^3\alpha_2 + 4a_0\sigma\omega_0\right) &= 0. \end{aligned} \tag{4.21}$$

For the nontrivial solution, the determinant of the coefficient matrix for Γ_1 and Γ_2 must vanish, which leads to a quadratic equation for the eigenvalue θ.

$$\theta = \frac{1}{4}(-a^2\xi + 2\alpha) \pm \sqrt{\frac{-3(3a^2\alpha_2 + 4\sigma_1\omega_o)a^2\alpha_2 + 4\alpha^2\omega_o^2}{16\omega_o^2}} \tag{4.22}$$

The solution is stable if and only if the real part of each of the eigenvalues of the coefficient of the matrix are less than or equal to zero.

5 SUBHARMONIC SOLUTION OF ORDER $(\frac{1}{3})$ $(\Omega \approx 3\omega_o)$

In this section, we study subharmonic solution of order $\frac{1}{3}$ i.e the periodic solution with period equal two multiple of the period of the excitation term i.e $(\Omega \approx 3\omega_o)$. Introducing the detuning parameter σ_2 to convert the small divisor term into secular term in Eq.(2.7).
i.e

$$\Omega = 3\omega_o + \epsilon\sigma_2, \tag{5.1}$$

so, we get

$$(\Omega - 2\omega_o)T_o = \omega_o T_o + t\sigma_2 T_o = \omega_o T_o + \sigma_2 T_1. \quad (5.2)$$

Eliminating the secular terms form the Eq.(5.2) yields

$$-2i\omega_o A' + i\alpha A\omega_o + 3A^2\alpha_2\bar{A} - i\xi A^2\omega_o\bar{A} - \frac{1}{4}(1 - \frac{1}{3}\bar{A})F_2\bar{A}^2 e^{i\sigma_2 T_1} = 0 \quad (5.3)$$

Using the polar form $A(T_1) = \frac{1}{2}a(T_1)e^{i\beta(T_1)}$, we obtain

$$\acute{a} = -(\frac{1}{16\omega_o}a^2F_2 - \frac{a^4F_2}{192\omega_o})\sin\psi - \frac{1}{8}a^3\xi + \frac{1}{2}a\alpha, \quad (5.4)$$

$$a\psi' = -(\frac{3}{16\omega_o}a^2F_2 - \frac{3a^4F_2}{192\omega_o})\cos\psi + \frac{9a^3\alpha_2}{8\omega_o} + a\sigma_2, \quad (5.5)$$

where a, ψ are the amplitude and the phase and $\psi = \sigma_2 T_1 - 3\beta$. for steady state solution, $\acute{a} = \acute{\psi} = 0$, in Eqs.(5.4) and (5.5) we obtain

$$(\frac{1}{16\omega_o}a_o^2F_2 - \frac{a_o^4F_2}{192\omega_o})\sin\psi_o = -\frac{1}{8}a_o^3\xi + \frac{1}{2}a_o\alpha. \quad (5.6)$$

$$(\frac{1}{16\omega_o}a_o^2F_2 - \frac{a_o^4F_2}{192\omega_o})\cos\psi_o = \frac{3a_o^3\alpha_2}{8\omega_o} + \frac{1}{3}a_o\sigma_2. \quad (5.7)$$

Equations (5.6)and(5.7)show that there are two possibilities: (trivial solution) at $a = 0$ and (nontrivial solution) at $a \neq 0$. Squaring and adding Eqs.(5.6)and(5.7) we get the frequency-response equation

$$\sigma_2 = \frac{-72a^2\alpha_2\omega_o \pm \sqrt{\Delta_1 F_2^2\omega_o^2 + \Delta_2\omega_o^4}}{64\omega_o^2}, \quad (5.8)$$

where

$\Delta_1 = 144a^2 - 24a^4 + a^6, \Delta_2 = -576a^4\xi^2 + 4608a^2\xi\alpha - 9216\alpha^2.$

Then, the first-order uniform expansion of the solution (first approximation) of Eq.(2.1) is given by

$$x = a\cos(\frac{1}{3}\Omega t - \frac{1}{3}\psi) + O(\epsilon) \quad (5.9)$$

Now, the analysis of the stability of the trivial solutions is determined as in the preceding section (4), so we get the eigenvalues equation obtained as

$$\theta_o = \pm\sqrt{\frac{\alpha^2}{4} - (\frac{\sigma_2}{2})^2}. \quad (5.10)$$

The solution is unstable if and only if the real part of the fixed points are positive. To determine the stability of the nontrivial solutions, we use the averaged Eq.(5.4) and Eq.(5.5) when the last term in these equations does not exist and let the nontrivial solutions have small variation from the steady state solutions a_o and γ_{20} so that

$$a(T_1) = a_0 + a_1(T_1) \qquad \& \qquad \gamma_2(T_1) = \gamma_{20} + \gamma_{22}(T_1), \quad (5.11)$$

where a_1 and γ_{11} are assumed to be infinitesimal. Thus, the solution of Eqs. (5.4) and (5.5) are stable or unstable depending on whether the functions a_1 and γ_{11} decay or grow with timeT_1. Inserting Eq.(5.11) into Eqs.(5.6) and (5.7) when the terms containing γ_2 in these equations does not exist and keeping only linear terms in the perturbed quantities, using steady-state Eqs.(5.6) and (5.7), we obtain

$$\begin{aligned} \acute{a_1} = & \frac{(36a_0^2 b\omega_0 + 3a_0^4 b\omega_0 + 144\alpha\omega_0 - 36a_0^2\alpha\omega_0)}{24\left(-12 + \text{a0}^2\right)\omega_0} a_1 \\ & + \frac{(108a_0^3\alpha_2 - 9a_0^5\alpha_2 + 96a_0\sigma\omega_0 - 8a_0^3\sigma\omega_0)}{24(-12 + a_0^2)\omega_0}\gamma_{22} \end{aligned} \tag{5.12}$$

$$\begin{aligned} \acute{\gamma_2} = & -\frac{3(36a_0^2\alpha_2 + 3a_0^4\alpha_2 - 32\sigma_2\omega_0 + 8a_0^2\sigma_2\omega_0)}{8(-12a_0 + a_0^3)\omega_0} a_1 \\ & + \frac{(-12a_0^3 b\omega_0 + a_0^5 b\omega_0 + 48a_0\alpha\omega_0 - 4a_0^3\alpha\omega_0)}{8(-12a_0 + a_0^3)\omega_0}\gamma_{22} \end{aligned} \tag{5.13}$$

Substituting $a_1 = \Gamma_1 e^{\theta T_1}$ and $\gamma_{11} = \Gamma_2 e^{\theta T_1}$ into Eq.(5.12) and Eq.(5.13), we get

$$\begin{array}{rcl} \Gamma_2(3a_0^5 b\omega_0 + 48a_0(3\alpha - 2\theta)\omega_0 - 4a_0^3(9b + 3\alpha - 2\theta)\omega_0) + & & \\ \Gamma_1(9a_0^2(12 + a_0^2)\alpha_2 - 96\sigma\omega_0 + 24a_0^2\sigma\omega_0) & = & 0 \\ \Gamma_1(-3a_0^4 b\omega_0 - 12a_0^2(3b - 3\alpha - 2\theta)\omega_0 - 144(\alpha + 2\theta)\omega_0) + & & \\ \Gamma_2(9a_0^3(-12 + a_0^2)\alpha_2 - 96a_0\sigma\omega_0 + 8a_0^3\sigma\omega_0) & = & 0 \end{array} \tag{5.14}$$

For the nontrivial solution, the determinant of the coefficient matrix for Γ_1 and Γ_2 must vanish, which leads to a quadratic equation for the eigenvalue θ.

$$\begin{aligned} \theta = & \frac{(24 - a^2)a^2\xi - 48\alpha}{8\Delta_3} \\ & \pm \frac{1}{8}\sqrt{(}\frac{1}{\Delta_3^2\omega_o^2}(\Delta_4\alpha_2^2 + \Delta_5\sigma_2\alpha_2\omega_o + (\Delta_6\xi^2 + \Delta_7\xi\alpha + \Delta_8\alpha^2 + \Delta_9\sigma_2)^2\omega_o^2)), \end{aligned} \tag{5.15}$$

where

$\Delta_3 = -12 + a^2, \Delta_4 = -3888a^4 + 27a^8, \Delta_5 = -1152a^4 + 96a^6,$
$\Delta_6 = 144a^4 - 48a^6 + 4a^8, \Delta_7 = -2304a^2 + 672a^4 - 48a^6,$
$\Delta_8 = 9216 - 2304a^2 + 144a^4$ and $\Delta_9 = 3072 - 1024a^2 + 64a^4$.

Consequently, a solution is stable if and only if the real parts of both eigenvalues Eq.(5.15) are less than or equal to zero.

6 NUMERICAL RESULTS AND DISCUSSIONS

By solving equations of the frequency response Eqs.(4.11) and (5.8) and stability conditions (4.17), (4.22), (5.10) and (5.15) numerically and plotting the numerical results in group of figures, we have Figs. (1-6) represent the frequency-response curves of the harmonic solution for the parameters $(\omega_o = 0.7, \xi = 10, h = 0.01, \alpha = -0.5, F_1 = 2, F_2 = 0.5, \alpha_2 = 0.2)$.

In Fig. (1), we observe the response amplitude has single-valued curve and all solutions are stable. The maximum point exist at $\sigma = 0$. For increasing and decreasing the damping factor α with negative value, we note that the single-valued curve shifts upward and downward so that the maximum value has increased and decreased magnitudes respectively, Fig. (2). When the coefficient of cubic term α_2 is decreased with negative value and increased with positive value, we note that the response amplitude is bent to the right and left and has harding and softing phenomena. The maximum points are exist at $\sigma = 1.024, \sigma = 0.67$ and $\sigma = 0.95$, Fig. (3). As the nature frequency ω_o take the values 0.7 and 5, we observe that the response amplitude shift upward and downward and has increased and decreased maximum value at $\sigma = 0$, Fig. (4). Fig. (5), when the coefficient of parametric excitation F_2 is decreased and increased with negative and positive values respectively, we get the same variation as in Fig. (2) and Fig. (6).

Figs. (7-13) represent the frequency-response curves of subharmonic solution of order ($\frac{1}{2}$) for the parameters $(\omega_o = 0.09, \xi = 0.1, h = 0.01, \alpha = 0.02, \alpha_2 = -0.04)$.

In Fig. (7), the response amplitude has multivalued curve so that the left branch has stable and unstable solution and the right branch has unstable solutions. There exist a harding phenomena because the multivalued curve is bent to the right. When $\alpha = 0.1$, we note that the multivaled curve is expanding and containing the main multivalued curve. The regions of multivalued, stability and stability are increased, Fig. (8). As $\alpha_2 = 0.04$, we note that the multivalued curve shifts to the left and the zones of multivalued and definition are decreased. For further increase of α_2, the multivalued curve shifts to the left and the zones of stability, multivalued and definition are increased. There exist a softing phenomena, Fig. (9). When α_2 is increased with negative value, the multivalued curve is expanded and shift to the right. The region of stability multivalued and definition are increased, Fig. (10). As $\omega_o = 0.1$, the multivalued curve shifts to right, Fig. (11). For increasing the coefficient of parametric excitation h, we observe the multivalued curve contracted and lies inside the main multivalued curve, Fig. (12). The regions of multivalued, stability and definition are decreased, Fig. (13).

Figs. (14-20) represent the frequency-response curves of subharmonic solution of order (one-to-third) for the parameters $(\omega_o = 0.9, \xi = 7, F_2 = 8, \alpha = 0.2, \alpha_2 = -10)$.

In Fig. (14), the response amplitude has oval which is bent to the right and there exist a harding phenomena. The left branch has stable and unstable solution and the right branch has unstable solutions. As α_2 is increased and decreased with negative values, we note that the oval shifts to the left and to the right so that the regions of multivalued, stability and definition are decreased and increased respectively, Fig. (15). When α_2 takes the values (5, 10, 15), we note that, the oval is bents to the left respectively and has softing phenomena, Fig. (16). For increasing α, we note that the oval expanding and contracted which lies outside and inside the main oval respectively. The regions of multivalued, stability and definition are increased and decreased respectively, Fig.(17). As $F_2 = 2$, the oval contracted and given a small oval lies in the main oval so that the zones of definition, multivalued and stability are decreased. When F_2 increased, the oval is expanded which containing the main oval so that the zones of definition, multivalued and stability are increased, Fig. (18). For decreases and increases ξ, we get the same variation as in Fig. (18)and Fig. (19). When ω_o takes the value (0.7, 1.7), we observe that the oval expanding and contracted and shifts to the downward and upward respectively. The zones of definition, multivalued and stability are increased and decreased respectively, Fig. (20).

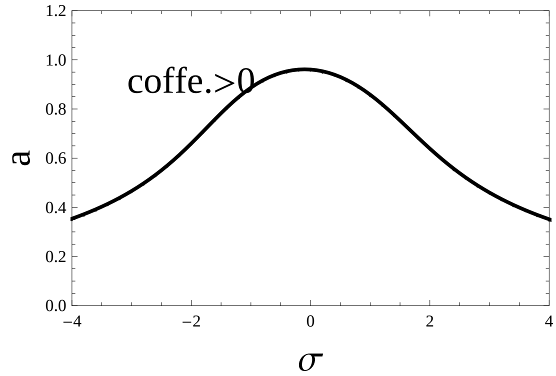

Fig. 1. The frequency response curves of the harmonic solution for the parameters ($\omega_o = 0.7, \xi = 10, F_1 = 2, F_2 = 0.5, \alpha = -0.5, \alpha_2 = 0.2$

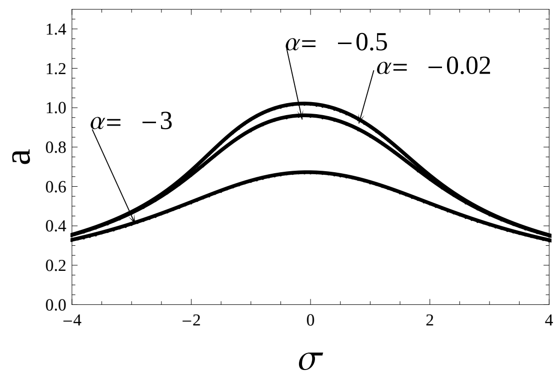

Fig. 2. Variation of the amplitude of the response with the detuning parameter for increasing and decreasing α

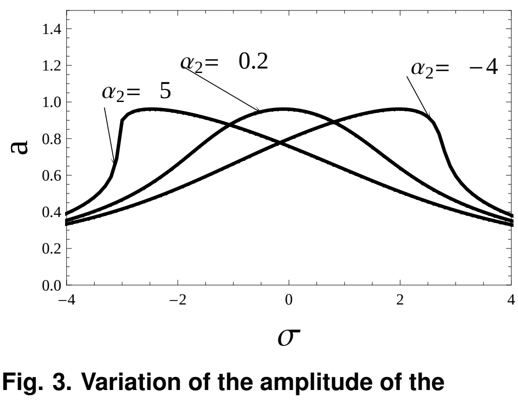

Fig. 3. Variation of the amplitude of the response with the detuning parameter for increasing and decreasing α_2

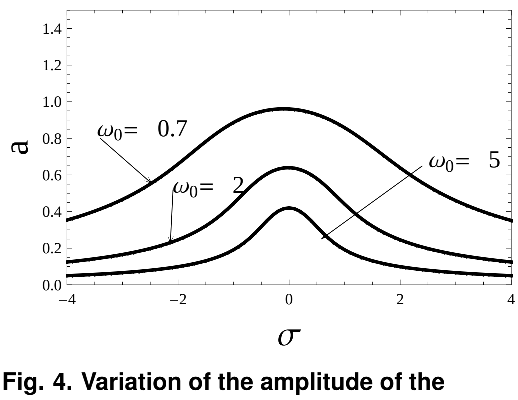

Fig. 4. Variation of the amplitude of the response with the detuning parameter for increasing and decreasing ω_o

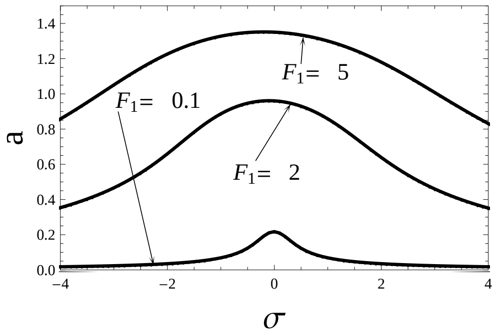

Fig. 5. Variation of the amplitude of the response with the detuning parameter for increasing and decreasing F_1

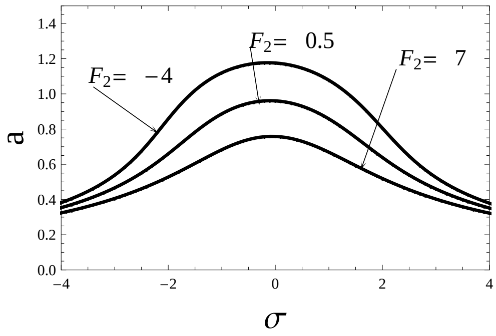

Fig. 6. Variation of the amplitude of the response with the detuning parameter for increasing and decreasing F_2

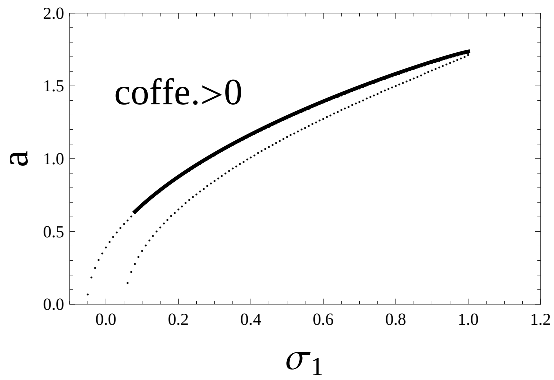

Fig. 7. The frequency response curves of the subharmonic solution of order $\frac{1}{2}$ for the parameters [$\omega_o = 0.09, \xi = 0.1, h = 0.01, \alpha = 0.02, \alpha_2 = -0.04$]

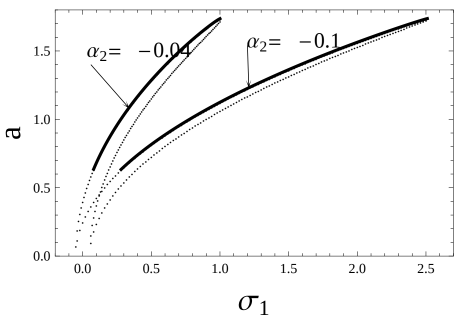

Fig. 8. Variation of the amplitude of the response with the detuning parameter for increasing α_2

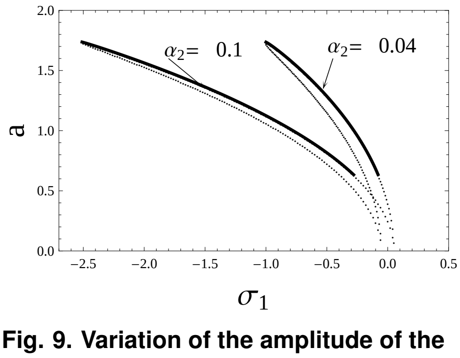

Fig. 9. Variation of the amplitude of the response with the detuning parameter for decreasing α_2

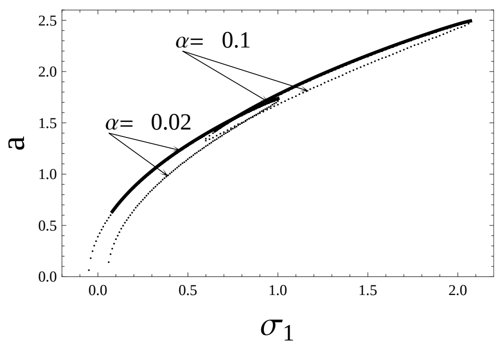

Fig. 10. Variation of the amplitude of the response with the detuning parameter for increasing and decreasing α

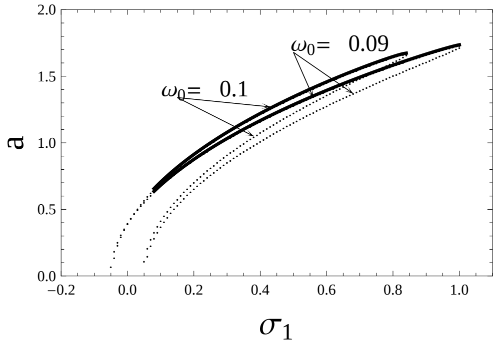

Fig. 11. Variation of the amplitude of the response with the detuning parameter for increasing and decreasing ω_o

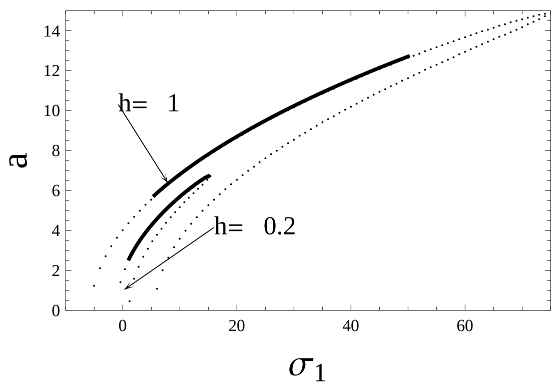

Fig. 12. Variation of the amplitude of the response with the detuning parameter for increasing and decreasing h

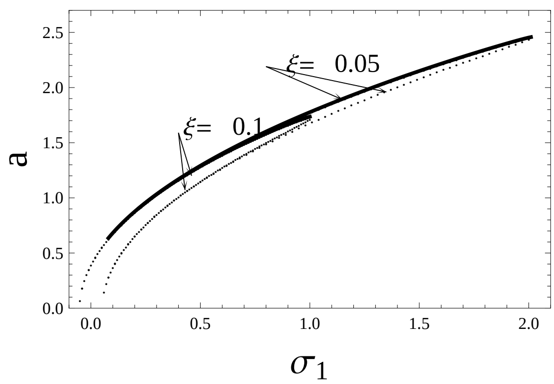

Fig. 13. Variation of the amplitude of the response with the detuning parameter for increasing and decreasing ξ

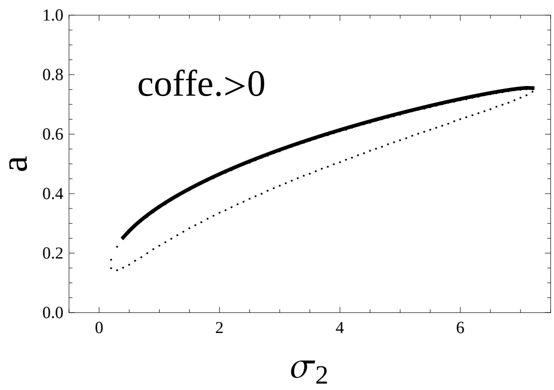

Fig. 14. The frequency response curves of subharmonic solution of order $\frac{1}{3}$ for the parameters [$\omega_o = 0.9, \xi = 7, F_2 = 8, \alpha = 0.2, \alpha_2 = -10$]

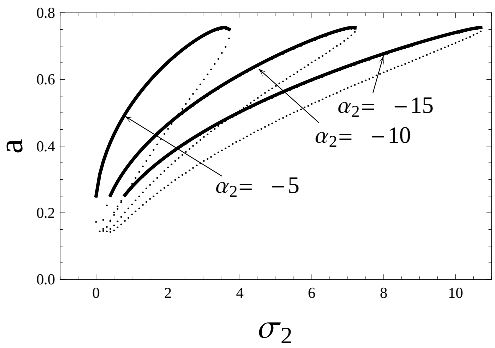

Fig. 15. Variation of the amplitude of the response with the detuning parameter for increasing α_2

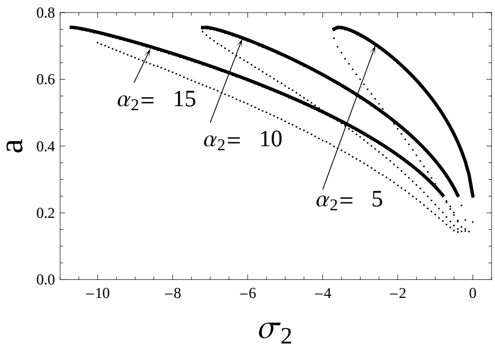

Fig. 16. Variation of the amplitude of the response with the detuning parameter for decreasing α_2

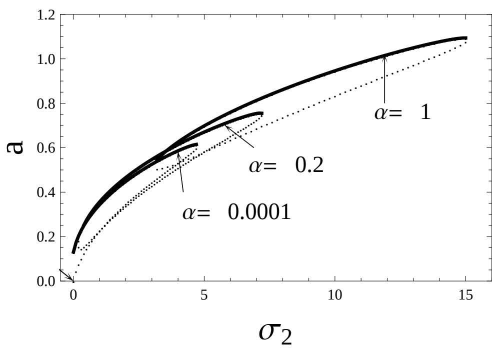

Fig. 17. Variation of the amplitude of the response with the detuning parameter for increasing and decreasing α

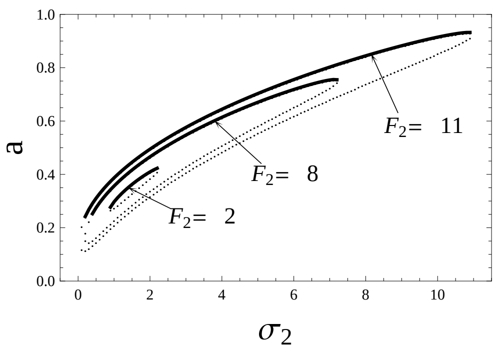

Fig. 18. Variation of the amplitude of the response with the detuning parameter for increasing and decreasing F_2

Out[37]=

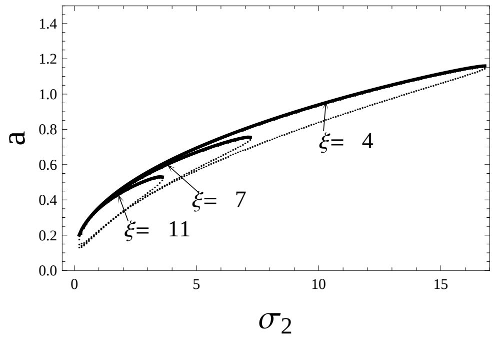

Fig. 19. Variation of the amplitude of the response with the detuning parameter for increasing and decreasing ξ

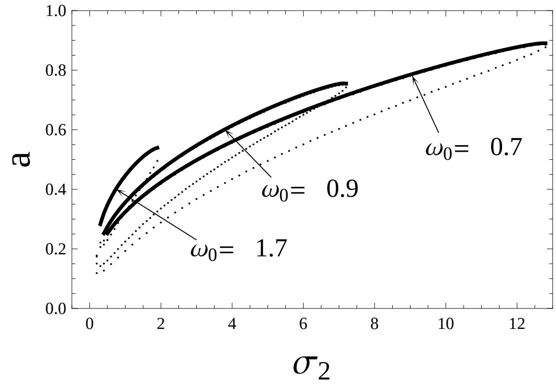

Fig. 20. Variation of the amplitude of the response with the detuning parameter for increasing and decreasing ω_o

7 CONCLUSION

In this paper, we have investigated an analytically of harmonic and subharmonic solutions of order $\frac{1}{2}$ and $\frac{1}{3}$ for a weakly nonlinear second order differential equation which governed modified Mathieu-van der PolDuffing equation. The method of multiple scales is used to determine the first approximate of the solution and two first order ordinary differential equations which describe the modulation of the amplitude and the phase. Steady state solution and its stability are obtained. Numerical solutions of the frequency response equation and the stability equation are carried out for different values of the parameters in the equation. Results are represented in a group of figures in which solid curves (dashed) are denoted stable (unstable) solutions.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

References

[1] Asghari M, Ahmadian MT, Kahrobaiyan MH, Rahaeifard M. On the size-dependent behavior of functionally graded micro-beams. Materials and Design. 2010;31:2324-2329.

[2] Rhoads JF, Shaw SW, Turner KL. Nonlinear dynamics and its applications in micro-and nanoresonators. J. Dyn. Syst. Meas. Control. 2010;132(3):034001, (14 pages).

[3] Sebald G, Kuwano H, Guyomar D, Ducharne B. Simulation of a Duffing os- cillator for broadband piezoelectric energy harvesting. Smart Mater. Struct. 2011;20(7).

[4] Marqus AF, Castell RC, Shkel AM. Modelling the electrostatic actuation of MEMS: State Of The Art 2005, Technical Report, IOC-DT. 2005;18:1-24.

[5] Rhoads JF, Shaw SW, Turner KL, Baskaran R. Tunable microelectromechanical Systems filters that exploit parametric resonance. Journal of Vibration and Acoustics. 2005;127(5):423-430.

[6] Rhoads JF, Shaw SW, Demartini BE, Moehlis J, Turner KL. Generalized parametric resonance in electrostatically actuated microelectromechanical oscil- lators. Journal of Sound and Vibration. 2006;296(4-5):797-829.

[7] Alsaleem FM, Younis MI, Ouakad HM. On the nonlinear resonances and dynamic pull-in of electrostatically actuated resonators. Journal of Micromechanics and Microengineering. 2009;19:1-14.

[8] Elnagger AM, El-Bassiouny AF, Mosa GA. Harmonic and sub-harmonic resonance of MEMS subjected to a weakly non-linear parametric and external excitations. International Journal of Applied Mathematical Research. 2013;2(2):252-263.

[9] Zhang W, Tang Y. Global dynamics of the cable under combined parametrical and ex- ternal excitations. International Journal of Non-Linear Mechanics. 2002;37(3):505526.

[10] Faravelli L, Ubertin F. Nonlinear state observation for cable dynamics. Journal of vibration and Control. 2009;15:10491077.

[11] Pandey M, Rand RH, Zehnder A. Perturbation analysis of entrainment in a micromechanical limit cycle oscillator. Commun. Nonlinear. Sci. Numer. Simul. 2007;12:12911301.

[12] Pandey M, Rand RH, Zehnder A. Frequency locking in a forced Mathieuvan der PolDuffing system. Nonlinear Dyn; 2007. DOI: 10.1007/s11071-007-9238-x

[13] Belhaq M, Houssni M. Quasi-periodic oscillations, chaos and suppression of chaos in a nonlinear oscillator driven by parametric and external excitations. Nonlinear Dyn. 1999;18:124.

[14] Rand RH, Guennoun K, Belhaq M. 2:2:1 Resonance in the quasi-periodic Mathieu equation. Nonlinear Dyn. 2003;31:187193.

[15] Mohamed Belhaq, Abdelhak Fahsi. 2:1 and 1:1 frequency-locking in fast excited van der PolMathieuDuffing oscillator. Nonlinear Dyn. 2008;53:139. DOI: 10.1007/s11071-007-9302-6

Peer-review history:
The peer review history for this paper can be accessed here:
http://sciencedomain.org/review-history/18720