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Perturbation Solutions to Fifth Order Over-damped Nonlinear Systems

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Abstract

Fifth order over-damp nonlinear differential systems can be used to describe many engineering problems and physical phenomena occur in the nature. In this article, the Krylov-Bogoliubov-Mitropolskii (KBM) method has been extended to investigate the solution of a certain fifth order over-damp nonlinear systems and desired result has been found. The implementation of the presented method is illustrated by an example. The first order analytical approximate solutions obtained by the method for different initial conditions show a good agreement with those obtains by numerical method.

Keywords: Non-linearity; over-damp; perturbation; eigenvalues; Runge-Kutta method.

AMS subject Classification: 34E05, 34E10.

1 Introduction

Many engineering problems and physical phenomena arise in the nature lead to over-damp nonlinear differential equations. There exist several approaches for analytical approximate solutions, such as, the Lindstedt-Poincare method [1], WKB method [2], Multi-time-scale method [3], the Krylov-Bogoliubov-Mitropolskii [4,5] method etc. The well-situated and widely used technique to obtain analytic approximate solutions to the nonlinear equations is the perturbation methods. Among the above methods KBM method is

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the mostly convenient and is the extensively used technique to obtain analytical approximate solution of nonlinear systems with small nonlinearity. In fact, Krylov and Bogoliubov [5] developed the perturbation method for obtaining periodic solutions was amplified and justified by Bogoliubov and Mitropolskii [4] and later Popov [6] and Meldelson [7] extended the method for damped nonlinear oscillations. A unified KBM method to solve second order nonlinear systems which covers under-damped, over-damped and periodic system with constant coefficients was presented by Murty [8]. Sattar [9] studied a third order over damped nonlinear system and Bojadziev [10] examined the damped oscillations modeled by a three dimensional nonlinear system. Shamsul and Sattar [11] presented a method for critically damped and Islam and Akbar [12] for more critically damped third order nonlinear systems. Akbar et al. [13] presented a method to solve fourth order over damped nonlinear systems which is easier, simple and less laborious than Murty et al. [14]. Later, Akbar et al. [15] pull out the method presented in [13] to damped oscillatory systems. Islam et al. [16] investigated the solutions of fourth order more critically damped nonlinear systems where Akbar [17] examined a different type solution for the same. Rahman et al. [18] obtained fourth order nonlinear oscillatory systems in which two of the eigenvalues are real and negative and the other two are complex conjugate. Akbar and Siddique [19] amplified the KBM method to obtain solutions of fifth order weakly nonlinear oscillatory systems. Also Optimal Homotopy Asymptotic Method [20-22] are developed for solving nonlinear evolutions equations.

The aim of this article is to obtain the analytical approximate solutions of fifth order over-damped nonlinear systems extending the KBM method. Figures are provided to compare validation and usefulness of the solutions obtained by the presented method for different initial conditions with the corresponding numerical solutions obtained by the fourth order Runge- Kutta method.

2 The Method

Let us consider a fifth order nonlinear over-damped system

$$\frac{d^5 x}{dt^5} + \sum_{i=1}^{4} c_i \frac{d^i x}{dt^i} + c_5 x = -\varepsilon f(x,t) \tag{1}$$

where ε is a small parameter, $f(x,t)$ is the nonlinear function, c_i, $i = 1,2,..,4$ are the characteristic parameters of the system defined by $c_1 = \sum_{i=1}^{5} \lambda_i$, $c_2 = \sum_{\substack{i,j=1 \\ i \neq j}}^{5} \lambda_i \lambda_j$, $c_3 = \sum_{\substack{i,j,k=1 \\ i \neq j \neq k}}^{5} \lambda_i \lambda_j \lambda_k$, $c_4 = \sum_{\substack{i,j,k,l=1 \\ i \neq j \neq k \neq l}}^{5} \lambda_i \lambda_j \lambda_k \lambda_l$ and $c_5 = \prod_{i=1}^{5} \lambda_i$ where $-\lambda_1, -\lambda_2, -\lambda_3, -\lambda_4, -\lambda_5$ are the eigenvalues of the unperturbed equation of (1).

When $\varepsilon = 0$, the unperturbed equation has the solution

$$x(t,0) = \sum_{j=1}^{5} a_{j,0} e^{-\lambda_j t} \tag{2}$$

where $a_{j,0}$, $j = 1,2....,5$ are arbitrary constants.

When $\varepsilon \neq 0$, we seek a solution in accordance with Shamsul [23] of the form

$$x(t,\varepsilon) = \sum_{j=1}^{5} a_j(t) e^{-\lambda_j t} + \varepsilon u_1(a_1, a_2, \cdots, a_5, t) + \cdots \tag{3}$$

where each $a_j; j=1,2,\cdots,5$, satisfies the equations

$$\dot{a}_j(t) = \varepsilon A_j(a_1, a_2, \cdots, a_5, t) + \cdots \tag{4}$$

Confining our attention to the first few terms $1,2,\cdots,m$ in the series expansions of equations (3) and (4), we calculate the functions u_1 and $A_j; j=1,2,\cdots,5$ such that $a_j; j=1,2,\cdots,5$, appearing in equation (3) and (4), satisfy the differential equation (1) with an accuracy of ε^{m+1}. Though the solution can be obtained up to the accuracy of any order of approximation, owing to the rapidly growing algebraic complexity for the derivation of the formulae, the solution in general confine to lower order [8]. In order to determine these unknown functions, it is assumed that the function u_1 exclude fundamental terms which are included in the series expansion (3) at order ε^0.

Differentiating $x(t,\varepsilon)$ five times with respect to t and substituting $x(t,\varepsilon)$ and their derivatives in eq. (1), using the relation in eq. (4) and equating the coefficients of ε, we obtain

$$\prod_{j=1}^{5}\left(\frac{d}{dt}+\lambda_j\right)u_1 + \sum_{j=1}^{5} e^{-\lambda_j t}\left(\prod_{k=1,\, j\neq k}^{5}\left(\frac{d}{dt}-\lambda_j+\lambda_k\right)\right)A_j = -f^{(0)}(a_1, a_2, \cdots, a_5, t) \tag{5}$$

where $f^{(0)} = f(x_0)$ and $x_0 = \sum_{j=1}^{5} a_j(t)e^{-\lambda_j t}$

The function $f^{(0)}$ can be expanded in a Taylor series (see Murty and Deekshatulu [24] for details) as:

$$f^{(0)} = \sum_{m_1=-\infty\cdots m_5=-\infty}^{\infty\ldots\ldots\infty} F_{m_1,\cdots m_5} \sum_{i=1}^{5} a_i^{m_i} e^{-(m_1\lambda_1+m_2\lambda_2+\cdots+m_5\lambda_5)t}$$

To obtain the solution of Eq. (1), it is assumed that u_1 exclude the fundamental terms. Therefore, Eq. (5) can be separated into six equations for unknown functions u_1 and $A_j; j=1,2,\ldots\ldots,5$ (see [23] for details).

Substituting the functional value and equating the coefficients of $e^{-\lambda_j t}; j=1,2,\ldots,5$, we obtain

$$e^{-\lambda_1 t}\sum_{i=2}^{5}\left(\frac{d}{dt}-\lambda_1+\lambda_i\right)A_1 = -\sum_{\substack{m_1=-\infty,\cdots,m_5=-\infty\\ m_3=m_4,\, m_1=m_2+1}}^{\infty\ldots\ldots\infty} F_{m_1,\cdots m_5}\sum_{i=1}^{5} a_i^{m_i} e^{-(m_1\lambda_1+m_2\lambda_2+\cdots+m_5\lambda_5)t} \tag{6}$$

$$e^{-\lambda_2 t}\sum_{i=1,\, i\neq 2}^{5}\left(\frac{d}{dt}-\lambda_2+\lambda_i\right)A_2 = -\sum_{\substack{m_1=-\infty,\cdots,m_5=-\infty\\ m_3=m_4,\, m_1=m_2-1}}^{\infty\ldots\ldots\infty} F_{m_1,\cdots m_5}\sum_{i=1}^{5} a_i^{m_i} e^{-(m_1\lambda_1+m_2\lambda_2+\cdots+m_5\lambda_5)t} \tag{7}$$

$$e^{-\lambda_3 t}\sum_{i=1,\, i\neq 3}^{5}\left(\frac{d}{dt}-\lambda_3+\lambda_i\right)A_3 = -\sum_{\substack{m_1=-\infty,\cdots,m_5=-\infty\\ m_1=m_2,\, m_3=m_4+1}}^{\infty\ldots\ldots\infty} F_{m_1,\cdots m_5}\sum_{i=1}^{5} a_i^{m_i} e^{-(m_1\lambda_1+m_2\lambda_2+\cdots+m_5\lambda_5)t} \tag{8}$$

$$e^{-\lambda_4 t}\sum_{i=1, i\neq 4}^{5}\left(\frac{d}{dt}-\lambda_4+\lambda_i\right)A_4=-\sum_{\substack{m_1=-\infty, m_5=-\infty \\ 1=m_2, m_3=m_4-1}}^{\infty.....\infty} F_{m_1,\cdots m_5}\sum_{i=1}^{5} a_i^{m_i} e^{-(m_1\lambda_1+m_2\lambda_2+.........+m_5\lambda_5)t} \tag{9}$$

$$e^{-\lambda_5 t}\sum_{i=1}^{4}\left(\frac{d}{dt}-\lambda_5+\lambda_i\right)A_5=-\sum_{\substack{m_1=-\infty\cdots m_5=-\infty \\ m_1=m_2, m_3=m_4}}^{\infty.....\infty} F_{m_1,\cdots m_5}\sum_{i=1}^{5} a_i^{m_i} e^{-(m_1\lambda_1+m_2\lambda_2+.........+m_5\lambda_5)t} \tag{10}$$

and

$$\sum_{i=1}^{5}\left(\frac{d}{dt}+\lambda_i\right)u_1=-\sum_{m_1=-\infty\cdots m_5=-\infty.}^{\infty.....\infty} F_{m_1,\cdots m_5}\sum_{i=1}^{5} a_i^{m_i} e^{-(m_1\lambda_1+m_2\lambda_2+.........+m_5\lambda_5)t} \tag{11}$$

where u_1 avoid the terms for $m_1=m_2\pm 1,\ m_3=m_4\pm 1,\ m_1=m_2,\ m_3=m_4$.

Solving Eqs. (6) to (11), we obtain the unknown functions $A_1, A_2,..., A_5$ and u_1.

It is possible to transform solution Eq. (3) to the exact formal KBM [14,15,19,23] solution by substituting $a_1=\frac{a}{2}e^{\varphi_1}$, $a_2=\frac{a}{2}e^{-\varphi_1}$, $a_3=\frac{b}{2}e^{\varphi_2}$, $a_4=\frac{b}{2}e^{-\varphi_2}$ and $a_5=c$. Herein a, b are amplitudes and φ_1, φ_2 are phase variables.

3 Example

As an example of the above procedure, we consider the Duffing type equation

$$\frac{d^5x}{dt^5}+\sum_{i=1}^{4} c_i\frac{d^ix}{dt^i}+c_5x=-\varepsilon x^3 \tag{12}$$

Here $f(x,t)=x^3$. Therefore, $f^{(0)}=(\sum_{i=1}^{5} a_i e^{-\lambda_i t})^3$

or

$$\begin{aligned}
f^{(0)} &= a_1^3e^{-3\lambda_1 t}+3a_1^2a_2e^{-(2\lambda_1+\lambda_2)t}+3a_1a_2^2e^{-(\lambda_1+2\lambda_2)t}+a_2^3e^{-3\lambda_2 t}\\
&+3a_1^2a_3e^{-(2\lambda_1+\lambda_3)t}+3a_1^2a_4e^{-(2\lambda_1+\lambda_4)t}+3a_1^2a_5e^{-(2\lambda_1+\lambda_5)t}\\
&+6a_1a_2a_3e^{-(\lambda_1+\lambda_2+\lambda_3)t}+6a_1a_2a_4e^{-(\lambda_1+\lambda_2+\lambda_4)t}+6a_1a_2a_5e^{-(\lambda_1+\lambda_2+\lambda_5)t}\\
&+3a_2^2a_3e^{-(2\lambda_2+\lambda_3)t}+3a_2^2a_4e^{-(2\lambda_2+\lambda_4)t}+3a_2^2a_5e^{-(2\lambda_2+\lambda_5)t}\\
&+3a_1a_3^2e^{-(\lambda_1+2\lambda_3)t}+3a_1a_4^2e^{-(\lambda_1+2\lambda_4)t}+3a_1a_5^2e^{-(\lambda_1+2\lambda_5)t}\\
&+6a_1a_3a_4e^{-(\lambda_1+\lambda_3+\lambda_4)t}+6a_1a_3a_5e^{-(\lambda_1+\lambda_3+\lambda_5)t}+6a_1a_4a_5e^{-(\lambda_1+\lambda_4+\lambda_5)t}\\
&+3a_2a_3^2e^{-(\lambda_2+2\lambda_3)t}+3a_2a_4^2e^{-(\lambda_2+2\lambda_4)t}+3a_2a_5^2e^{-(\lambda_2+2\lambda_5)t}\\
&+6a_2a_3a_4e^{-(\lambda_2+\lambda_3+\lambda_4)t}+6a_2a_4a_5e^{-(\lambda_2+\lambda_4+\lambda_5)t}+6a_2a_3a_5e^{-(\lambda_2+\lambda_3+\lambda_5)t}\\
&+a_3^3e^{-3\lambda_3 t}+3a_3^2a_4e^{-(2\lambda_3+\lambda_4)t}+3a_3a_4^2e^{-(\lambda_3+2\lambda_4)t}+a_4^3e^{-3\lambda_4 t}\\
&+3a_3^2a_5e^{-(2\lambda_3+\lambda_5)t}+6a_3a_4a_5e^{-(\lambda_3+\lambda_4+\lambda_5)t}+3a_4^2a_5e^{-(2\lambda_4+\lambda_5)t}\\
&+3a_3a_5^2e^{-(\lambda_3+2\lambda_5)t}+3a_4a_5^2e^{-(\lambda_4+2\lambda_5)t}+3a_5^3e^{-3\lambda_5 t}
\end{aligned} \tag{13}$$

Thus the equations (6) to (11) takes the form

$$e^{-\lambda_1 t}\sum_{i=2}^{5}(\frac{d}{dt}-\lambda_1+\lambda_i)A_1=-3a_1^2a_2e^{-(2\lambda_1+\lambda_2)t}-6a_1a_3a_4e^{-(\lambda_1+\lambda_3+\lambda_4)t} \tag{14}$$

$$e^{-\lambda_2 t}\sum_{i=1,i\neq 2}^{5}(\frac{d}{dt}-\lambda_2+\lambda_i)A_2=-3a_1a_2^2e^{-(\lambda_1+2\lambda_2)t}-6a_2a_3a_4e^{-(\lambda_2+\lambda_3+\lambda_4)t} \tag{15}$$

$$e^{-\lambda_3 t}\sum_{i=1,i\neq 3}^{5}(\frac{d}{dt}-\lambda_3+\lambda_i)A_3=-3a_3^2a_4e^{-(2\lambda_3+\lambda_4)t}-6a_1a_2a_3e^{-(\lambda_1+\lambda_2+\lambda_3)t} \tag{16}$$

$$e^{-\lambda_4 t}\sum_{i=1,i\neq 4}^{5}(\frac{d}{dt}-\lambda_4+\lambda_i)A_4=-3a_3a_4^2e^{-(\lambda_3+2\lambda_4)t}-6a_1a_2a_4e^{-(\lambda_1+\lambda_2+\lambda_4)t} \tag{17}$$

$$e^{-\lambda_5 t}\sum_{i=1}^{4}(\frac{d}{dt}-\lambda_5+\lambda_i)A_5=-6a_1a_2a_5e^{-(\lambda_1+\lambda_2+\lambda_5)t}-6a_3a_4a_5e^{-(\lambda_3+\lambda_4+\lambda_5)t} \tag{18}$$

and

$$\begin{aligned}\sum_{i=1}^{5}(\frac{d}{dt}+\lambda_i)u_1=&-(a_1^3e^{-3\lambda_1 t}+a_2^3e^{-3\lambda_2 t}+3a_1^2a_3e^{-(2\lambda_1+\lambda_3)t}+3a_1^2a_4e^{-(2\lambda_1+\lambda_4)t}+3a_1^2a_5e^{-(2\lambda_1+\lambda_5)t}\\&+3a_2^2a_3e^{-(2\lambda_2+\lambda_3)t}+3a_2^2a_4e^{-(2\lambda_2+\lambda_4)t}+3a_2^2a_5e^{-(2\lambda_2+\lambda_5)t}\\&+3a_1a_3^2e^{-(\lambda_1+2\lambda_3)t}+3a_1a_4^2e^{-(\lambda_1+2\lambda_4)t}+3a_1a_5^2e^{-(\lambda_1+2\lambda_5)t}\\&+6a_1a_3a_5e^{-(\lambda_1+\lambda_3+\lambda_5)t}+6a_1a_4a_5e^{-(\lambda_1+\lambda_4+\lambda_5)t}\\&+3a_2a_3^2e^{-(\lambda_2+2\lambda_3)t}+3a_2a_4^2e^{-(\lambda_2+2\lambda_4)t}+3a_2a_5^2e^{-(\lambda_2+2\lambda_5)t}\\&+6a_2a_4a_5e^{-(\lambda_2+\lambda_4+\lambda_5)t}+6a_2a_3a_5e^{-(\lambda_2+\lambda_3+\lambda_5)t}\\&+a_3^3e^{-3\lambda_3 t}+a_4^3e^{-3\lambda_4 t}+3a_3^2a_5e^{-(2\lambda_3+\lambda_5)t}+3a_4^2a_5e^{-(2\lambda_4+\lambda_5)t}\\&+3a_3a_5^2e^{-(\lambda_3+2\lambda_5)t}+3a_4a_5^2e^{-(\lambda_4+2\lambda_5)t}+3a_5^3e^{-3\lambda_5 t})\end{aligned} \tag{19}$$

Solving Eqs. (14) to (18) and inserting $\lambda_1=k_1-\omega_1$, $\lambda_2=k_1+\omega_1$, $\lambda_3=k_2-\omega_2$, $\lambda_4=k_2+\omega_2$ and $\lambda_5=\xi$, we obtain

$$\begin{aligned}A_1=&-\frac{3a_1^2a_2e^{-2k_1t}}{2(k_1-\omega_1)\{(3k_1-k_2)-(\omega_1-\omega_2)\}\{(3k_1-k_2)-(\omega_1+\omega_2)\}\{(3k_1-\xi)-\omega_1\}}\\&-\frac{6a_1a_3a_4e^{-2k_2t}}{2(k_2-\omega_1)\{(k_1+k_2)-(\omega_1-\omega_2)\}\{(k_1+k_2)-(\omega_1+\omega_2)\}\{(k_1+2k_2-\xi)-\omega_1\}}\end{aligned}$$

$$\begin{aligned}A_2=&-\frac{3a_1a_2^2e^{-2k_1t}}{2(k_1+\omega_1)\{(3k_1-k_2)+(\omega_1+\omega_2)\}\{(3k_1-k_2)+(\omega_1-\omega_2)\}\{(3k_1-\xi)+\omega_1\}}\\&-\frac{6a_2a_3a_4e^{-2k_2t}}{2(k_2+\omega_1)\{(k_1+k_2)+(\omega_1+\omega_2)\}\{(k_1+k_2)+(\omega_1-\omega_2)\}\{(k_1+2k_2-\xi)+\omega_1\}}\end{aligned}$$

$$A_3=-\frac{3a_3^2a_4e^{-2k_2t}}{2(k_2-\omega_2)\{(3k_2-k_1)+(\omega_1-\omega_2)\}\{(3k_2-k_1)-(\omega_1+\omega_2)\}\{(3k_2-\xi)-\omega_2\}}$$
$$-\frac{6a_1a_3a_4e^{-2k_1t}}{2(k_1-\omega_2)\{(k_1+k_2)+(\omega_1-\omega_2)\}\{(k_1+k_2)-(\omega_1+\omega_2)\}\{(2k_1+k_2-\xi)-\omega_2\}}$$
$$A_4=-\frac{3a_3a_4^2e^{-2k_2t}}{2(k_2+\omega_2)\{(3k_2-k_1)+(\omega_1+\omega_2)\}\{(3k_2-k_1)-(\omega_1-\omega_2)\}\{(3k_2-\xi)+\omega_2\}}$$
$$-\frac{6a_1a_2a_4e^{-2k_1t}}{2(k_1+\omega_2)\{(k_1+k_2)+(\omega_1+\omega_2)\}\{(k_1+k_2)-(\omega_1-\omega_2)\}\{(2k_1+k_2-\xi)+\omega_2\}}$$
$$A_5=-\frac{6a_1a_2a_5e^{-2k_1t}}{\{(k_1+\xi)^2-\omega_1^2)\}\{(2k_1-k_2+\xi)^2-\omega_2^2\}}-\frac{6a_3a_4a_5e^{-2k_2t}}{\{(k_2+\xi)^2-\omega_2^2)\}\{(2k_2-k_1+\xi)^2-\omega_2^2\}}$$

Now inserting $A_j;\, j=1,2,..,5$ in the Eq. (4) and using $a_1=\frac{1}{2}ae^{\varphi_1}$, $a_2=\frac{1}{2}ae^{-\varphi_1}$, $a_3=\frac{1}{2}be^{\varphi_2}$, $a_4=\frac{1}{2}be^{-\varphi_2}$ and $a_5=c$ we obtain

$$\begin{aligned}
&\dot a=\varepsilon(l_1a^3e^{-2k_1t}+l_2ab^2e^{-2k_2t}) && \dot b=\varepsilon(n_1b^3e^{-2k_2t}+n_2a^2be^{-2k_1t})\\
&\dot\varphi_1=\varepsilon(m_1a^2e^{-2k_1t}+m_2b^2e^{-2k_2t}) && \dot\varphi_2=\varepsilon(p_1b^2e^{-2k_2t}+p_2a^2e^{-2k_1t})\\
&\dot c=\varepsilon(q_1a^2ce^{-2k_1t}+q_2b^2ce^{-2k_2t})
\end{aligned}\tag{20}$$

where

$$l_1=-\frac{3}{8}\left\{\frac{\begin{array}{c}(3k_1^2-k_1k_2+\omega_1^2+\omega_1\omega_2)(9k_1^2-3k_1k_2-3k_1\xi+k_2\xi+\omega_1^2-\omega_1\omega_2)+\\(4k_1\omega_1+k_1\omega_2-k_2\omega_1)(6k_1\omega_1-k_2\omega_1-3k_1\omega_2-\omega_1\xi+\omega_2\xi)\end{array}}{(k_1^2-\omega_1^2)\{(3k_1-k_2)^2-(\omega_1+\omega_2)^2\}\{(3k_1-k_2)^2-(\omega_1-\omega_2)^2\}\{(3k_1-\xi)^2-\omega_1^2\}}\right\}$$

$$l_2=-\frac{3}{4}\left\{\frac{\begin{array}{c}(k_1k_2+k_2^2+\omega_1^2-\omega_1\omega_2)\{(k_1+k_2)(k_1+2k_2-\xi)+\omega_1^2-\omega_1\omega_2\}+\\(2k_2\omega_1-k_2\omega_2+k_1\omega_1)(2k_1\omega_1+3k_2\omega_1+k_1\omega_2+2k_2\omega_2-\omega_1\xi-\omega_2\xi)\end{array}}{(k_2^2-\omega_1^2)\{(k_1+k_2)^2-(\omega_1-\omega_2)^2\}\{(k_1+k_2)^2-(\omega_1+\omega_2)^2\}\{(k_1+2k_2-\xi)^2-\omega_1^2\}}\right\}$$

$$m_1=-\frac{3}{8}\left\{\frac{\begin{array}{c}(3k_2^2-k_1k_2+\omega_2^2+\omega_1\omega_2)(9k_2^2-3k_1k_2-3k_2\xi+k_1\xi+\omega_2^2-\omega_1\omega_2)+\\(4k_2\omega_2-k_1\omega_2+k_2\omega_1)(6k_2\omega_2-k_1\omega_2-3k_2\omega_1+\omega_1\xi-\omega_2\xi)\end{array}}{(k_2^2-\omega_2^2)\{(3k_2-k_1)^2-(\omega_1-\omega_2)^2\}\{(3k_2-k_1)^2-(\omega_1+\omega_2)^2\}\{(3k_2-\xi)^2-\omega_2^2\}}\right\}$$

$$m_2=-\frac{3}{4}\left\{\frac{\begin{array}{c}(k_1^2+k_1k_2+\omega_2^2+\omega_1\omega_2)\{(k_1+k_2)(2k_1+k_2-\xi)+\omega_2^2-\omega_1\omega_2\}+\\(k_1\omega_1+2k_1\omega_2+k_2\omega_2)(2k_2\omega_2+3k_1\omega_2-2k_1\omega_1-k_2\omega_1+\omega_1\xi-\omega_2\xi)\end{array}}{(k_1^2-\omega_2^2)\{(k_2+k_1)^2-(\omega_1-\omega_2)^2\}\{(k_2+k_1)^2-(\omega_1+\omega_2)^2\}\{(2k_1+k_2-\xi)^2-\omega_2^2\}}\right\}$$

$$n_1=-\frac{3}{8}\left\{\frac{\begin{array}{c}(3k_1^2-k_1k_2+\omega_1^2+\omega_1\omega_2)(6k_1\omega_1-k_2\omega_1-3k_1\omega_2-\omega_1\xi+\omega_2\xi)+\\(4k_1\omega_1+k_1\omega_2-k_2\omega_1)(9k_1^2-3k_1k_2-3k_1\xi+k_2\xi+\omega_1^2-\omega_1\omega_2)\end{array}}{(k_1^2-\omega_1^2)\{(3k_1-k_2)^2-(\omega_1+\omega_2)^2\}\{(3k_1-k_2)^2-(\omega_1-\omega_2)^2\}\{(3k_1-\xi)^2-\omega_1^2\}}\right\}$$

$$n_2 = -\frac{3}{4}\left\{\frac{\begin{array}{c}(k_1k_2 + k_2^2 + \omega_1^2 - \omega_1\omega_2)(2k_1\omega_1 + 3k_2\omega_1 + k_1\omega_2 + 2k_2\omega_2 - \omega_1\xi - \omega_2\xi) - \\ (2k_2\omega_1 - k_2\omega_2 + k_1\omega_1)\{(k_1 + k_2)(k_1 + 2k_2 - \xi) + \omega_1^2 + \omega_1\omega_2\}\end{array}}{(k_2^2 - \omega_1^2)\{(k_1 + k_2)^2 - (\omega_1 - \omega_2)^2\}\{(k_1 + k_2)^2 - (\omega_1 + \omega_2)^2\}\{(k_1 + 2k_2 - \xi)^2 - \omega_1^2\}}\right\}$$

$$p_1 = -\frac{3}{8}\left\{\frac{\begin{array}{c}(3k_2^2 - k_1k_2 + \omega_2^2 + \omega_1\omega_2)(6k_2\omega_2 - k_1\omega_2 - 3k_2\omega_1 + \omega_1\xi - \omega_2\xi) - \\ (4k_2\omega_2 - k_1\omega_2 + k_2\omega_1)(9k_2^2 - 3k_1k_2 - 3k_2\xi + k_1\xi + \omega_2^2 - \omega_1\omega_2)\end{array}}{(k_2^2 - \omega_2^2)\{(3k_2 - k_1)^2 - (\omega_1 - \omega_2)^2\}\{(3k_2 - k_1)^2 - (\omega_1 + \omega_2)^2\}\{(3k_2 - \xi)^2 - \omega_2^2\}}\right\}$$

$$p_2 = -\frac{3}{4}\left\{\frac{\begin{array}{c}(k_1^2 + k_1k_2 + \omega_2^2 + \omega_1\omega_2)(2k_2\omega_2 + 3k_1\omega_2 - 2k_1\omega_1 - k_2\omega_1 + \omega_1\xi - \omega_2\xi) \\ - (k_1\omega_1 + 2k_1\omega_2 + k_2\omega_2)\{(k_1 + k_2)(2k_1 + k_2 - \xi) + \omega_2^2 - \omega_1\omega_2\}\end{array}}{(k_1^2 - \omega_2^2)\{(k_2 + k_1)^2 - (\omega_1 - \omega_2)^2\}\{(k_2 + k_1)^2 - (\omega_1 + \omega_2)^2\}\{(2k_1 + k_2 - \xi)^2 - \omega_2^2\}}\right\}$$

$$q_1 = -\frac{3}{2}\frac{1}{\{(k_1 + \xi)^2 - \omega_1^2)\}\{(2k_1 - k_2 + \xi)^2 - \omega_2^2\}}$$

and

$$q_2 = -\frac{3}{2}\frac{1}{\{(k_2 + \xi)^2 - \omega_2^2)\}\{(2k_2 - k_1 + \xi)^2 - \omega_2^2\}}$$

Equations in (20) are nonlinear and have no exact solutions. We can solve (20) by considering a, b, c, φ_1 and φ_2 are constants in the right-hand sides of (20). Since ε is small, $\dot{a}, \dot{b}, \dot{c}, \dot{\varphi}_1$ and $\dot{\varphi}_2$ are slowly varying function of time, therefore, this consideration is applicable. This assumption was used by Murty et al. [13,24] to solve the similar nonlinear equations. The solution is thus

$$\begin{aligned}
a(t) &= a_0 + \varepsilon(l_1 a_0{}^3 \frac{(1 - e^{-2k_1 t})}{2k_1} + l_2 a_0 b_0{}^2 \frac{(1 - e^{-2k_2 t})}{2k_2}) \\
b(t) &= b_0 + \varepsilon(n_1 b_0{}^3 \frac{(1 - e^{-2k_2 t})}{2k_2} + n_2 a_0{}^2 b_0 \frac{(1 - e^{-2k_1 t})}{2k_1}) \\
\varphi_1(t) &= \varphi_{1,0} + \varepsilon(m_1 a_0{}^2 \frac{(1 - e^{-2k_1 t})}{2k_1} + m_2 b_0{}^2 \frac{(1 - e^{-2k_2 t})}{2k_2}) \\
\varphi_2(t) &= \varphi_{2,0} + \varepsilon(p_1 b_0{}^2 \frac{(1 - e^{-2k_2 t})}{2k_2} + p_2 a_0{}^2 \frac{(1 - e^{-2k_1 t})}{2k_1})
\end{aligned}$$

and

$$c(t) = c_0 + \varepsilon(q_1 a_0{}^2 c_0 \frac{(1 - e^{-2k_1 t})}{2k_1} + q_2 b_0{}^2 c_0 \frac{(1 - e^{-2k_2 t})}{2k_2}) \tag{21}$$

Therefore, the first order solution of Eq. (12) is

$$x(t) = a\cosh(\omega_1 t + \varphi_1) + b\cosh(\omega_2 t + \varphi_2) + ce^{-\xi t} + \varepsilon u_1 . \tag{22}$$

where a, b, c, φ_1 and φ_2 are given in the Eq. (21).

4 Results and Discussion

In order to check the accuracy of an analytical approximate solutions obtained, based on the KBM method, we compare the approximate solutions to the numerical solutions. In this article, we have compared our obtained results to those obtained by the fourth order Runge-Kutta method for different sets of initial conditions as well as different sets of eigenvalues. Beside this, we have also computed the Pearson correlation between the perturbation results and the corresponding numerical results. From the figures we observed that our perturbation solution agree with numerical results suitably for different initial conditions.

At first, for $k_1 = 2.05$, $k_2 = 3.09$, $\omega_1 = 1.414, \omega_2 = 2.57$, $\xi = 0.03$ and $\varepsilon = 0.1$, $x(t,\varepsilon)$ has been computed by solution (22), in which a, b, c, φ_1 and φ_2 are computed by the equation (21) with initial conditions $a_0 = 0.63$, $b_0 = 0.52$, $c_0 = 0.3$, $\varphi_{1\,0} = 1.375$ and $\varphi_{2\,0} = 0.5708$ [i. e, $x(0) = 2.23253$, $\frac{dx(0)}{dt} = -2.144589$, $\frac{d^2x(0)}{dt^2} = 6.275333$, $\frac{d^3x(0)}{dt^3} = -30.274223$ and $\frac{d^4x(0)}{dt^4} = 162.2777256$.]

For the above mentioned initial conditions, the perturbation results obtained by the solution (22) and the corresponding numerical results obtained by a fourth order Runge-Kutta method with a small time increment $\Delta t = 0.05$, are plotted in Fig. 1. The correlation between the results is 0.994941.

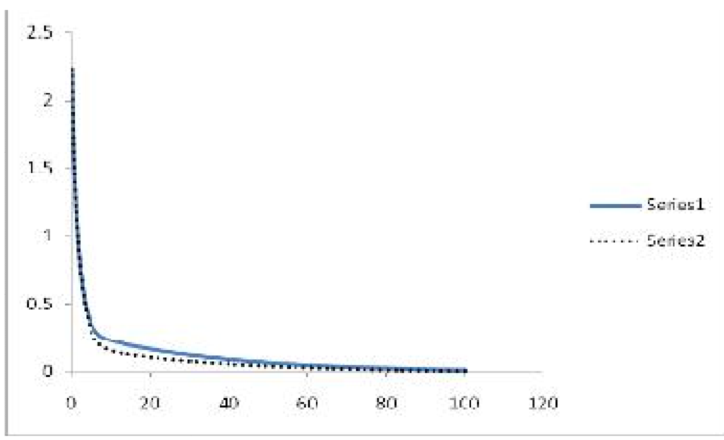

Fig. 1. Perturbation results are plotted by solid line and numerical results plotted dotted line

Secondly, for $k_1 = 1$, $k_2 = 2$, $\omega_1 = 0.81$, $\omega_2 = 1.5704$, $\xi = 0.09$ and $\varepsilon = 0.1$, $x(t,\varepsilon)$ has been computed (22), in which a, b, c, φ_1 and φ_2 by the equation (21)with initial conditions $a_0 = 0.15, b_0 = 0.15, c_0 = 0.20, \varphi_{1,0} = 1.25664$ and $\varphi_{2,0} = 0.3927$ [i. e., $x(0) = 0.646581$, $\frac{dx(0)}{dt} = -0.334486$, $\frac{d^2x(0)}{dt^2} = 0.745605$, $\frac{d^3x(0)}{dt^3} = -2.439806$ and $\frac{d^4x(0)}{dt^4} = 8.461052$.]

In this section, the perturbation results obtained by the solution (22) and the corresponding numerical results obtained by a fourth order Runge-Kutta method with a small time increment $\Delta t = 0.05$, are plotted Fig. 2. The correlation between the results is 0.994337.

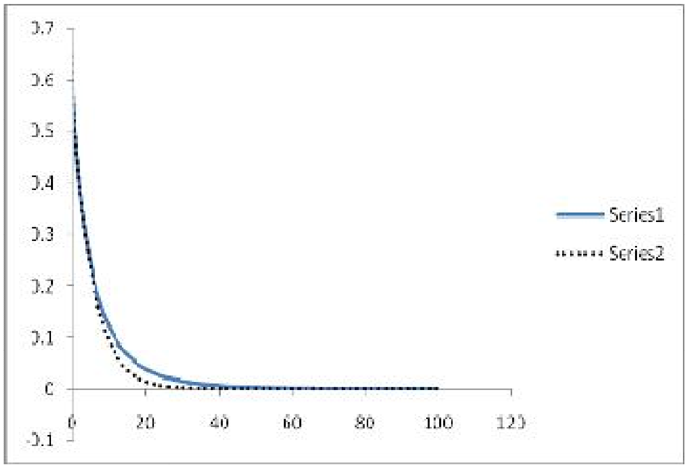

Fig. 2. Perturbation solution plotted by solid line and numerical solution plotted dotted line

Finally, for $k_1 = 0.5, k_2 = 0.47, \omega_1 = 0.237, \omega_2 = 0.321, \xi = 0.003$ and $\varepsilon = 0.1$, $x(t,\varepsilon)$ has been computed (22), in which a, b, c, φ_1 and φ_2 by the equation (21) with initial conditions $a_0 = 0.02, b_0 = 0.02, c_0 = 0.008, \varphi_{1,0} = .\pi/2$ and $\varphi_{2,0} = 0.414$ [i. e. $x(0) = 0.079922$, $\frac{dx(0)}{dt} = -0.021774$, $\frac{d^2x(0)}{dt^2} = 0.009020$, $\frac{d^3x(0)}{dt^3} = -0.005127$ and $\frac{d^4x(0)}{dt^4} = 0.003545$.]

The perturbation results obtained by the solution (22) and the corresponding numerical results obtained by a fourth order Runge-Kutta method with a small time increment $\Delta t = 0.05$ are plotted Fig. 3. The correlation between the results is 0.999248.

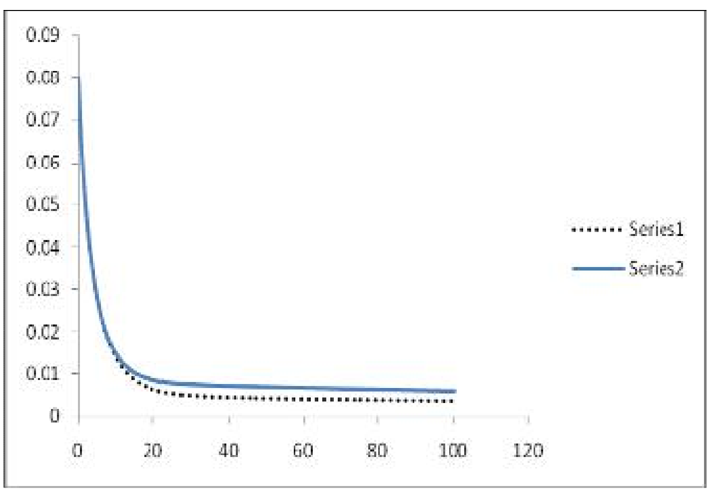

Fig. 3. Perturbation solution plotted by solid line and numerical solution plotted dotted line

From Figs. 1 to 3 it is noteworthy to observe that perturbation results show a good agreement with those obtained by the fourth order Runge-Kutta method.

5 Conclusions

In this article a procedure is formulated to find the first order analytical approximate solution of fifth order over damped nonlinear differential systems with small nonlinearities based on the KBM [4,5] method. The correlation has been calculated between the results acquired by the perturbation solution and the fourth order Runge-Kutta method of the same problem. The results obtained for different initial conditions, show a good coincidence with corresponding numerical results and they are strongly correlated.

Competing Interests

Authors have declared that no competing interests exist.

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