Asian Research Journal of Mathematics

16(10): 137-143, 2020; Article no.ARJOM.62033

ISSN: 2456-477X

Subordination Results for a Class of Analytic Functions Defined by Convolution

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/ARJOM/2020/v16i1030235

Editor(s):

(1) Dr. Sheng Zhang, Bohai University, China.

Reviewers:

(1) Fateme Ghomanjani, Iran.

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(3) Dheyaa Jasim Kadhim, University of Baghdad, Iraq.

Complete Peer review History: http://www.sdiarticle4.com/review-history/62033

Received: 28 August 2020

Accepted: 02 November 2020

Published: 20 November 2020

Original Research Article

Abstract

Aims/ Objectives: In this paper, making use the Hadamard product , we introduce drive several interesting subordination results for a new class of analytic function. Furthermore, we mention some known and new results, which follow as special cases of our results.

Keywords: Analytic functions; convolution; subordination; factor sequence.

2010 Mathematics Subject Classification: 30C45.

1 Introduction

Let $\mathbb{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : \ |z| < 1\}$. Let K denote the class of functions $f(z) \in \mathbb{A}$ which are convex in $\mathbb{U}$ and let $S(k, \alpha)$ denote the subclass of $\mathbb{A}$ which satisfies the following

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inequality (see [1, 2, 3, 4])

$$Re\left(\frac{zf^{'}(z)}{f(z)}+k\frac{z^{2}f^{''}(z)}{f(z)}\right)>\alpha \quad (k\geq 0; 0\leq \alpha <1; z\in \mathbb{U})\,.$$

The Hadamard product (or convolution) $(f*g)(z)$ of the functions $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z)=z+\sum_{n=2}^{\infty}b_{n}z^{n} \qquad (b_{n}\geq 0)\,, \tag{1.2}$$

is defined by:

$$(f*g)(z)=z+\sum_{n=2}^{\infty}a_{n}b_{n}z^{n}=(g*f)(z). \tag{1.3}$$

If f and g are analytic functions in $\mathbb{U}$, we say that f is subordinate to g, written $f\prec g$ if there exists a Schwarz function w, which (by definition) is analytic in $\mathbb{U}$ with $w(0)=0$ and $|w(z)|<1$ for all $z\in\mathbb{U}$, such that $f(z)=g(w(z)),\ z\in\mathbb{U}$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence (cf., e.g., [5, 6]):

$$f(z)\prec g(z)\Leftrightarrow f(0)=g(0) \text{ and } f(\mathbb{U})\subset g(\mathbb{U}).$$

Definition 1.1. (Subordinating Factor Sequence) *[7].* A sequence $\{d_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if, whenever f of the form (1.1) is analytic, univalent and convex in $\mathbb{U}$, we have the subordination given by

$$\sum_{n=1}^{\infty}d_{n}a_{n}z^{n}\prec f(z) \quad (z\in\mathbb{U};\ a_1=1). \tag{1.4}$$

For $0\leq\alpha<1, k\geq 0$ and for all $z\in\mathbb{U}$, let $H(f,g;\alpha,k)$ denote the subclass of $\mathbb{A}$ consisting of functions $f(z)$ of the form (1.1) and $g(z)$ of the form (1.2) and satisfying the analytic criterion:

$$Re\left\{\frac{z(f*g)^{'}(z)}{(f*g)(z)}+k\frac{z^{2}\left(f*g\right)^{''}(z)}{(f*g)(z)}\right\}>\alpha. \tag{1.5}$$

The class was introduce and studied by Aouf et al. (see [8]). We note that for suitable choice of g, we obtain the following subclasses studied by various authors.

(1) If we take $g(z)=\frac{z}{1-z}$, then the class $H(f,\frac{z}{1-z};\alpha,k)$ reduces to the class $S(k,\alpha)$ (see [3]);
(2) If we take $g(z)=\frac{z}{1-z}$ and $k=0$, then the class $H(f,\frac{z}{1-z};\alpha,0)$ reduces to the class $S^{*}(\alpha)$ (see [9]);
(3) If we take $g(z)=\frac{z}{(1-z)^2}$ and $k=0$, then the class $H(f,\frac{z}{(1-z)^2};\alpha,0)$ reduces to the class $K(\alpha)$ (see [9]);
(4) If we take

$$g(z)=z+\sum_{n=2}^{\infty}\sigma_{n}z^{n} \tag{1.6}$$

($or\ \ b_n=\sigma_n$), where

$$\sigma_{n}=\frac{\Theta\Gamma(\alpha_{1}+A_{1}(n-1)).......\Gamma(\alpha_{q}+A_{q}(n-1))}{(n-1)!\Gamma(\beta_{1}+B_{1}(n-1)).......\Gamma(\beta_{s}+B_{s}(n-1))} \tag{1.7}$$

$$(\alpha_i, A_i>0, i=1,.....q; \beta_j, B_j>0, j=1,...,s; q\leq s+1; q,s\in\mathbb{N}, \mathbb{N}=\{1,2,3,....\})$$

and

$$\Theta = \frac{\left(\prod_{j=0}^{s}\Gamma\left(\beta_j\right)\right)}{\left(\prod_{i=0}^{q}\Gamma\left(\alpha_i\right)\right)}, \tag{1.8}$$

then the class $H(f, z + \sum_{n=2}^{\infty} \sigma_n z^n; \alpha, k)$ reduces to the class $W_s^q(\alpha, k)$ (see [10])

$$= \left\{ f \in \mathbb{A} : Re\left\{ \frac{z\left(W_s^q f(z)\right)'}{W_s^q f(z)} + k\frac{z^2\left(W_s^q f(z)\right)''}{W_s^q f(z)} \right\} > \alpha,\ 0 \leq \alpha < 1; k \geq 0; q, s \in \mathbb{N}; z \in \mathbb{U} \right\}, \tag{1.9}$$

where $W_s^q f(z)$ is the Wright's generalized hypergeometric function (see [11, 12]) which contains well known operators such as the Dziok-Srivastava operator (see [13, 14]), the Carlson-Shaffer linear operator (see [15]), the Bernardi-Libera-Livingston operator (see [16, 17, 18]), Srivastava - Owa fractional derivative operator (see [19]), the Ruscheweyh derivative operator (see [20]) and the Noor integral operator of n-th order (see [21]);

(4) If we take

$$g(z) = z + \sum_{n=2}^{\infty} \left(\frac{l + 1 + \mu(n-1)}{l+1} \right)^m z^n \tag{1.10}$$

$(or\ b_n = \left(\frac{l+1+\mu(n-1)}{l+1}\right)^m,\ m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mu \geq 0, l \geq 0)$, then the class $H(f, z + \sum_{n=2}^{\infty} \left(\frac{l+1+\mu(n-1)}{l+1}\right)^m z^n;$ $\alpha, k)$ reduces to the class ${}_m(\mu, l, \alpha, k)$ (see [8]):

$$= \left\{ f \in \mathbb{A} : Re\left\{ \frac{z(I^m(\mu,l)f(z))'}{I^m(\mu,l)f(z)} + k\frac{z^2(I^m(\mu,l)f(z))''}{I^m(\mu,l)f(z)} \right\} > \alpha,\ 0 \leq \alpha < 1;\ k \geq 0;\ m \in \mathbb{N}_0;\ \mu,\ l \geq 0,\ z \in \mathbb{U} \right\}, \tag{1.11}$$

where $I^m(\gamma, l)f(z)$ is the extended multiplier transformation (see [22]), for $l = 0$,$\gamma \geq 0$, the operator $I_m(\gamma, 0) = D_\gamma^m$ was introduced and studied by Al-Oboudi (see [23]) and for $l = \gamma = 0$, the operator $I_m(0, 0) = D^m$, where D^m is Salagean differential operator see. [24].

2 Main Results

Unless otherwise mentioned, we shall assume in the reminder of this paper that, $0 \leq \alpha < 1,\ k \geq 0,\ n \geq 2,\ z \in \mathbb{U}$ *and* $g(z)$ is defined by (1.2). To prove our main results we shall need the following lemmas.

Lemma 2.1. *[7]. The sequence* $\{d_n\}_{n=1}^{\infty}$ *is a subordinating factor sequence if and only if*

$$Re\left\{ 1 + 2\sum_{n=1}^{\infty} d_n z^n \right\} > 0\ , \quad (z \in \mathbb{U}). \tag{2.1}$$

Lemma 2.2. *[8]. Let the function* $f\left(z\right)$ *defined by (1.1) satisfy the following condition:*

$$\sum_{n=2}^{\infty} \left(kn^2 + n - kn - \alpha\right) b_n \left|a_n\right| \leq 1 - \alpha. \tag{2.2}$$

Then $f\ (z) \in H(f, g; \alpha, k)$.

Let $H^*(f,g;\alpha,k)$ denote the class of functions $f(z) \in \mathbb{A}$ whose coefficients satisfy the condition (2.2). We note that $H^*(f,g;\alpha,k) \subseteq H(f,g;\alpha,k), S^*(k,\alpha) \subseteq S(k,\alpha), W_s^{*q}(\alpha,k) \subseteq W_s^q(\alpha,k)$ and $\overset{*}{}_m(\mu,l,\alpha,k) \subseteq_m (\mu,l,\alpha,k)$.

Theorem 2.3. *Let $f \in H^*(f,g;\alpha,k)$, $b_n \geq b_2 > 0$ $(n \geq 2)$. Then for every convex function $\phi \in K$, we have*

$$\frac{(2k-\alpha+2)\,b_2}{2[(2k-\alpha+2)\,b_2+(1-\alpha)]}(f*\phi)(z) \prec \phi(z), \tag{2.3}$$

and

$$Re\{f(z)\} > -\frac{(2k-\alpha+2)\,b_2+(1-\alpha)}{(2k-\alpha+2)\,b_2}. \tag{2.4}$$

The constant $\dfrac{(2k-\alpha+2)\,b_2}{2[(2k-\alpha+2)\,b_2+(1-\alpha)]}$ *is the best estimate.*

Proof. □

Let $f(z) \in H^*(f,g;\alpha,k)$ and let $\phi(z) = z + \sum_{n=2}^{\infty} c_n z^n \in K$. Then we have

$$\begin{aligned} &\frac{(2k-\alpha+2)\,b_2}{2[(2k-\alpha+2)\,b_2+(1-\alpha)]}(f*\phi)(z) \\ =\ &\frac{(2k-\alpha+2)\,b_2}{2[(2k-\alpha+2)\,b_2+(1-\alpha)]}\left(z+\sum_{n=2}^{\infty} a_n c_n z^n\right). \end{aligned} \tag{2.5}$$

Thus, by Definition 1, the subordination result (2.3) will hold true if the sequence

$$\left\{\frac{(2k-\alpha+2)\,b_2}{2[(2k-\alpha+2)\,b_2+(1-\alpha)]}a_n\right\}_{n=1}^{\infty}, \tag{2.6}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1, this is equivalent to the following inequality:

$$Re\left\{1+\sum_{n=1}^{\infty}\frac{(2k-\alpha+2)\,b_2}{[(2k-\alpha+2)\,b_2+(1-\alpha)]}a_n z^n\right\} > 0. \tag{2.7}$$

Now, since

$$\left(kn^2+n-kn-\alpha\right)b_n,$$

is an increasing function of n $(n \geq 2)$, we have

$$\begin{aligned} &Re\left\{1+\sum_{n=1}^{\infty}\frac{(2k-\alpha+2)\,b_2}{(2k-\alpha+2)\,b_2+(1-\alpha)}a_n z^n\right\} \\ =\ &Re\left\{1+\frac{(2k-\alpha+2)\,b_2}{(2k-\alpha+2)\,b_2+(1-\alpha)}z \right. \\ &\left. +\frac{1}{(2k-\alpha+2)\,b_2+(1-\alpha)}\sum_{n=2}^{\infty}(2k-\alpha+2)\,b_2 a_n z^n\right\} \\ \geq\ &1-\frac{(2k-\alpha+2)\,b_2}{(2k-\alpha+2)\,b_2+(1-\alpha)}r \\ &-\left(\frac{1}{(2k-\alpha+2)\,b_2+(1-\alpha)}\sum_{n=2}^{\infty}\left(kn^2+n-kn-\alpha\right)b_n\,|a_n|\,r^n\right) \end{aligned}$$

$$\begin{aligned} &> 1-\frac{(2k-\alpha+2)\,b_2}{(2k-\alpha+2)\,b_2+(1-\alpha)}r-\frac{(1-\alpha)}{(2k-\alpha+2)\,b_2+(1-\alpha)}r \\ &= 1-r>0\ (|z|=r<1), \end{aligned}$$

where we have also made use of assertion (2.2) of Lemma 2. Thus (2.7) holds true in $\mathbb{U}$. This proves the inequality (2.3). The inequality (2.4) follows from (2.3) by taking the convex function $\phi(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n \in K$. To prove the sharpness of the constant $\dfrac{(2k-\alpha+2)\,b_2}{2\,[(2k-\alpha+2)\,b_2+(1-\alpha)]}$, we consider the function $f_0(z) \in H^*(f, g; \alpha, k)$ given by

$$f_0(z) = z - \frac{(1-\alpha)}{(2k-\alpha+2)\,b_2} z^2. \tag{2.8}$$

Thus from (2.3), we have

$$\frac{(2k-\alpha+2)\,b_2}{2\,[(2k-\alpha+2)\,b_2+(1-\alpha)]} f_0(z) \prec \frac{z}{1-z}. \tag{2.9}$$

Moreover, it can easily be verified for the function $f_0(z)$ given by (2.8) that

$$\min_{|z|\le r}\left\{ Re\frac{(2k-\alpha+2)\,b_2}{2\,[(2k-\alpha+2)\,b_2+(1-\alpha)]} f_0(z)\right\} = -\frac{1}{2}. \tag{2.10}$$

This show that the constant $\dfrac{(2k-\alpha+2)\,b_2}{2\,[(2k-\alpha+2)\,b_2+(1-\alpha)]}$ is the best possible. This completes the proof of Theorem 1.

Putting $g(z) = z + \sum_{n=2}^{\infty} \sigma_n z^n$, where σ_n is defined by (1.7), in Lemma 2 and Theorem 1, we obtain the following corollary:

Corollary 2.4. *Let f defined by (1.1) be in the class $W_s^{*q}(\alpha, k)$ and satisfy the condition*

$$\sum_{n=2}^{\infty} \left(kn^2 + n - kn - \alpha\right) \sigma_n\, |a_n| \le 1 - \alpha.$$

Then for every function $\phi \in K$, we have

$$\frac{(2k-\alpha+2)\,\sigma_2}{2[(2k-\alpha+2)\,\sigma_2+(1-\alpha)]}(f * \phi)(z) \prec \phi(z),$$

and

$$Re\{f(z)\} > -\frac{(2k-\alpha+2)\,\sigma_2+(1-\alpha)}{(2k-\alpha+2)\,\sigma_2}.$$

The constant $\dfrac{(2k-\alpha+2)\,\sigma_2}{2[(2k-\alpha+2)\,\sigma_2+(1-\alpha)]}$ *is the best estimate.*

Putting $g(z) = z + \sum_{n=2}^{\infty} \left(\frac{l+1+\mu(n-1)}{l+1}\right)^m z^n$, in Lemma 2 and Theorem 1, we obtain the following corollary:

Corollary 2.5. *Let f defined by (1.1) be in the class $^*_m(\mu, l, \alpha, k)$ and satisfy the condition*

$$\sum_{n=2}^{\infty} \left(kn^2 + n - kn - \alpha\right) \left(\frac{l+1+\mu(n-1)}{l+1}\right)^m |a_n| \le 1 - \alpha.$$

Then for every function $\phi \in K$, we have

$$\frac{(2k-\alpha+2)\left(\frac{l+1+\mu}{l+1}\right)^m}{2[(2k-\alpha+2)\left(\frac{l+1+\mu}{l+1}\right)^m+(1-\alpha)]}(f*\phi)(z) \prec \phi(z),$$

and

$$Re\{f(z)\} > -\frac{(2k-\alpha+2)\left(\frac{l+1+\mu}{l+1}\right)^m+(1-\alpha)}{(2k-\alpha+2)\left(\frac{l+1+\mu}{l+1}\right)^m}.$$

The constant $\dfrac{(2k-\alpha+2)\left(\frac{l+1+\mu}{l+1}\right)^m}{2[(2k-\alpha+2)\left(\frac{l+1+\mu}{l+1}\right)^m+(1-\alpha)]}$ *is the best estimate.*

Remark 2.1. (i) Putting $g(z)=\frac{z}{1-z}$ (or $b_n=1$) and $k=0$ in Theorem 1, we obtain the result obtained by Frasin [[25], Corollary 2.3];

(ii) Putting $g(z)=\frac{z}{(1-z)^2}$ *and* $k=0$ *in Theorem 1,we obtain the result obtained by Frasin [[25],* Corollary 2.6].

3 Conclusions

In this work we presented some new drive several interesting subordination results for a new class of analytic function defined by convolution. This theorem leave open several possibilities that are worth investigating.

Acknowledgements

The author would like to thank the referees of the paper for their helpful suggestions.

Competing Interests

The author declare that there are no conflicts of interest regarding the publication of this paper.

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