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A Study on Generalized Tetranacci Numbers: Closed Form Formulas $\sum_{k=0}^{n} x^k W_k^2$ of Sums of the Squares of Terms

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Authors contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

In this paper, closed forms of the sum formulas $\sum_{k=0}^{n} x^k W_k^2$ for the squares of generalized Tetranacci numbers are presented. We also present the sum formulas $\sum_{k=0}^{n} x^k W_{k+1} W_k$, $\sum_{k=0}^{n} x^k W_{k+2} W_k$, and $\sum_{k=0}^{n} x^k W_{k+3} W_k$. As special cases, we give summation formulas of the of Tetranacci, Tetranacci-Lucas and some other fourth order linear recurrance sequences.

Keywords: Sum of squares; fourth order recurrence; Tetranacci numbers; Tetranacci-Lucas numbers.

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1 Introduction

There have been so many studies of the sequences of numbers in the literature which are defined recursively. Two of these type of sequences are the sequences of Tetranacci and Tetranacci-Lucas which are special case of generalized Tetranacci numbers. A generalized Tetranacci sequence

$$\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n\geq 0}$$

is defined by the fourth-order recurrence relations

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4}, \qquad W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3, \quad n \geq 4 \quad (1)$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3$ not all being zero.

This sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [1,2,3,4,5,6].

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{t}{u}W_{-(n-1)} - \frac{s}{u}W_{-(n-2)} - \frac{r}{u}W_{-(n-3)} + \frac{1}{u}W_{-(n-4)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1) holds for all integer n.

In literature, for example, the following names and notations (see Table 1) are used for the special case of r, s, t, u and initial values.

Table 1. A few special case of generalized Tetranacci sequences

Sequences (Numbers)	Notation	OEIS [7]
Tetranacci	$\{M_n\} = \{W_n(0,1,1,2;1,1,1,1)\}$	A000078
Tetranacci-Lucas	$\{R_n\} = \{W_n(4,1,3,7;1,1,1,1)\}$	A073817
fourth order Pell	$\{P_n^{(4)}\} = \{W_n(0,1,2,5;2,1,1,1)\}$	A103142
fourth order Pell-Lucas	$\{Q_n^{(4)}\} = \{W_n(4,2,6,17;2,1,1,1)\}$	A331413
modified fourth order Pell	$\{E_n^{(4)}\} = \{W_n(0,1,1,3;2,1,1,1)\}$	A190139
fourth order Jacobsthal	$\{J_n^{(4)}\} = \{W_n(0,1,1,1;1,1,1,2)\}$	A007909
fourth order Jacobsthal-Lucas	$\{j_n^{(4)}\} = \{W_n(2,1,5,10;1,1,1,2)\}$	A226309
modified fourth order Jacobsthal	$\{K_n^{(4)}\} = \{W_n(3,1,3,10;1,1,1,2)\}$	
4-primes	$\{G_n\} = \{W_n(0,0,1,2;2,3,5,7)\}$	
Lucas 4-primes	$\{H_n\} = \{W_n(4,2,10,41;2,3,5,7)\}$	
modified 4-primes	$\{E_n\} = \{W_n(0,0,1,1;2,3,5,7)\}$	

Here OEIS stands for On-line Encyclopedia of Integer Sequences. In the rest of the paper, for easy writing, we drop the superscripts and write $P_n, Q_n, E_n, J_n, j_n, K_n$ and for $P_n^{(4)}, Q_n^{(4)}, E_n^{(4)}, J_n^{(4)}, j_n^{(4)}, K_n^{(4)}$, respectively. For generalized fourth order Pell numbers and generalized 4-primes numbers see [8] and [9], respectively.

The evaluation of sums of powers of these sequences is a challenging issue. Two pretty examples are

$$\sum_{k=0}^{n}(-1)^k M_k^2 = (-1)^n\left(-M_{n+2}^2 + M_{n+1}^2 + M_{n+4}M_{n+2} - M_{n+3}M_{n+2} - M_{n+3}M_{n+1}\right)$$

and

$$\sum_{k=0}^{n}(-1)^k R_k^2 = (-1)^n\left(-R_{n+2}^2 + R_{n+1}^2 + R_{n+4}R_{n+2} - R_{n+3}R_{n+2} - R_{n+3}R_{n+1}\right) + 7.$$

In this work, we derive expressions for sums of second powers of generalized Tetranacci numbers. We present some works on sum formulas of powers of the numbers in the following Table 2.

Table 2. A few special study on sum formulas of second, third and arbitrary powers

Name of sequence	sums of second powers	sums of third powers	sums of powers
Generalized Fibonacci	[10,11,12,13,14,15,16,17,18]	[19,20,21,22,23,24,25]	[26,27,28,29]
Generalized Tribonacci	[30,31,32]		
Generalized Tetranacci	[33,34,35]		

2 An Application of the Sum of the Squares of the Numbers

An application of the sum of the squares of the numbers is circulant matrix. Computations of the Frobenius norm, spectral norm, maximum column length norm and maximum row length norm of circulant (r-circulant, geometric circulant, semicirculant) matrices with the generalized m-step Fibonacci sequences require the sum of the squares of the numbers of the sequences. For generalized m-step Fibonacci sequences see for example Soykan [36]. If $m = 2, m = 3$ and $m = 4$, we get the generalized Fibonacci sequence, generalized Tribonacci sequence and generalized Tetranacci sequence, respectively. Next, we recall some information on circulant (r-circulant, geometric circulant) matrices and Frobenius norm, spectral norm, maximum column length norm and maximum row length norm.

Circulant matrices have been around for a long time and have been extensively used in many scientific areas. In some scientific areas such as image processing, coding theory and signal processing we often encounter circulant matrices. These matrices also have many applications in numerical analysis, optimization, digital image processing, mathematical statistics and modern technology.

Let $n \geq 2$ be an integer and r be any real or complex number. An $n \times n$ matrix C_r is called a r-circulant matrix if it of the form

$$C_r = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ rc_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ rc_1 & rc_2 & rc_3 & \cdots & rc_{n-1} & c_0 \end{pmatrix}_{n\times n}.$$

and the r-circulant matrix C_r is denoted by $C_r = Circ_r(c_0, c_1, ..., c_{n-1})$. If $r = 1$ then 1-circulant matrix is called as circulant matrix and denoted by $C = Circ(c_0, c_1, ..., c_{n-1})$. Circulant matrix was first proposed by Davis in [37]. This matrix has many interesting properties, and it is one of the most important research subject in the field of the computational and pure mathematics (see for example references given in Table 3). For instance, Shen and Cen [38] studied on the norms of r-circulant matrices with Fibonacci and Lucas numbers. Then, later Kızılateş and Tuglu [39] defined a new geometric circulant matrix as follows:

$$C_{r^*} = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ r^2c_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ r^{n-1}c_1 & r^{n-2}c_2 & r^{n-3}c_3 & \cdots & rc_{n-1} & c_0 \end{pmatrix}_{n\times n}.$$

and then they obtained the bounds for the spectral norms of geometric circulant matrices with the generalized Fibonacci number and Lucas numbers. When the parameter satisfies $r = 1$, we get the classical circulant matrix. See also Polatlı [40] for the spectral norms of r-circulant matrices with a type of Catalan triangle numbers.

The Frobenius (or Euclidean) norm and spectral norm of a matrix $A = (a_{ij})_{m\times n} \in M_{m\times n}(\mathbb{C})$ are defined respectively as follows:

$$\|A\|_F = \left(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\right)^{1/2} \quad \text{and} \quad \|A\|_2 = \left(\max_{1\le i\le n}|\lambda_i|\right)^{1/2}$$

where λ_i 's are the eigenvalues of the matrix A^*A and A^* is the conjugate of transpose of the matrix A . The maximum column length norm $c_1(.)$ and the maximum row length norm $r_1(.)$ of an matrix of order $n \times n$ are defined as follows:

$$c_1(A) = \max_{1\le j\le n}\left(\sum_{i=1}^{n}|a_{ij}|^2\right)^{1/2} \quad \text{and} \quad r_1(A) = \max_{1\le i\le n}\left(\sum_{j=1}^{n}|a_{ij}|^2\right)^{1/2}.$$

The following inequality holds for any matrix $A = M_{n\times n}(\mathbb{C})$:

$$\frac{1}{\sqrt{n}}\|A\|_F \le \|A\|_2 \le \|A\|_F .$$

Calculations of the above norms $\|A\|_F$, $\|A\|_2$, $c_1(A)$ and $r_1(A)$ require the sum of the squares of the numbers a_{ij}. As in our case, the numbers a_{ij} can be chosen as elements of second, third or higher order linear recurrence sequences.

In the following Table 3, we present a few special study on the Frobenius norm, spectral norm, maximum column length norm and maximum row length norm of circulant (r-circulant, geometric circulant, semicirculant) matrices with the generalized m-step Fibonacci sequences which require sum formulas of second powers of numbers in m-step Fibonacci sequences ($m = 2, 3, 4$).

Table 3. Papers on the norms

Name of sequence	Papers
second order↓	second order↓
Fibonacci, Lucas	[41,42,43,39,44,45,46,47,48,38,49,50,51,52]
Pell, Pell-Lucas	[53,54]
Jacobsthal, Jacobsthal-Lucas	[55,56,57,58]
third order↓	third order↓
Tribonacci, Tribonacci-Lucas	[59,30,60]
Padovan, Perrin	[61,62,63]
fourth order↓	fourth order↓
Tetranacci, Tetranacci-Lucas	[64]

Also linear summing formulas of the generalized m-step Fibonacci sequences are required for the computation of various norms of circulant matrices circulant matrices with the generalized m-step Fibonacci sequences. We present some works on summing formulas of the numbers in the following Table 4.

Table 4. A few special study of sum formulas

Name of sequence	Papers which deal with summing formulas
Pell and Pell-Lucas	[65,66,67],[68,69]
Generalized Fibonacci	[70,71,72,73,74,75,76]
Generalized Tribonacci	[77,78,79]
Generalized Tetranacci	[80,81,6]
Generalized Pentanacci	[82,83]
Generalized Hexanacci	[84,85]

3 Main Result

Let

$$\Delta = (ux^2 - t^2x^3 - u^2x^4 - u^3x^6 + sx + rtx^2 + 2sux^3 + r^2ux^3 + su^2x^5 - rtux^4 + 1)$$
$$(r^2x + 2ux^2 - s^2x^2 + t^2x^3 - u^2x^4 + 2sx + 2rtx^2 - 2sux^3 - 1).$$

Theorem 3.1. *If $\Delta \neq 0$ then*

(a)

$$\sum_{k=0}^{n} x^k W_k^2 = \frac{\Delta_1}{\Delta},$$

(b)

$$\sum_{k=0}^{n} x^k W_{k+1} W_k = \frac{\Delta_2}{\Delta},$$

(c)

$$\sum_{k=0}^{n} x^k W_{k+2} W_k = \frac{\Delta_3}{\Delta},$$

(d)

$$\sum_{k=0}^{n} x^k W_{k+3} W_k = \frac{\Delta_4}{\Delta},$$

where

$$\Delta_1 = \sum_{k=1}^{20} \Gamma_k, \ \Delta_2 = \sum_{k=1}^{20} \Theta_k, \ \Delta_3 = \sum_{k=1}^{20} \Phi_k, \ \Delta_4 = \sum_{k=1}^{20} \Psi_k$$

with

$\Gamma_1 = -x^{n+4}(ux^2 + t^2x^3 + u^2x^4 - u^3x^6 + sx + rtx^2 + r^2ux^3 - su^2x^5 + rtux^4 - 1)W_{n+4}^2,$

$\Gamma_2 = -x^{n+3}(r^2x + ux^2 + t^2x^3 + u^2x^4 - u^3x^6 + sx + r^2t^2x^4 + r^2u^2x^5 - r^2u^3x^7 + rtx^2 + r^2sx^2 + r^3tx^3 + r^4ux^4 - su^2x^5 - r^2su^2x^6 + 2rstx^3 + 3rtux^4 + 2r^2sux^4 - 2rtu^2x^6 + r^3tux^5 - 1)W_{n+3}^2,$

$\Gamma_3 = -x^{n+2}(r^2x + ux^2 + s^2x^2 - s^3x^3 + t^2x^3 + u^2x^4 - u^3x^6 + sx + r^2t^2x^4 + r^2u^2x^5 + s^2t^2x^5 - r^2u^3x^7 - s^2u^2x^6 - s^3u^2x^7 - s^2u^3x^8 + rtx^2 + r^2sx^2 + r^3tx^3 + r^4ux^4 + s^2ux^4 - su^2x^5 - 2s^3ux^5 - r^2s^2ux^5 - r^2su^2x^6 + 4rstx^3 + 3rtux^4 - rs^2tx^4 + 4r^2sux^4 - 2rtu^2x^6 + r^3tux^5 + 2st^2ux^6 + rs^2tux^6 + 2rstu^2x^7 - 1)W_{n+2}^2,$

$\Gamma_4 = -x^{n+1}(r^2x + ux^2 + s^2x^2 - s^3x^3 + 2t^2x^3 + u^2x^4 - t^4x^6 - u^3x^6 + sx + r^2t^2x^4 + r^2u^2x^5 + s^2t^2x^5 - r^2u^3x^7 - s^2u^2x^6 - s^3u^2x^7 - s^2u^3x^8 + t^2u^2x^7 - t^2u^3x^9 + rtx^2 + r^2sx^2 + r^3tx^3 - rt^3x^5 - st^2x^4 + r^4ux^4 + s^2ux^4 - su^2x^5 - 2s^3ux^5 - t^2ux^5 - r^2s^2ux^5 - r^2su^2x^6 - r^2t^2ux^6 + st^2u^2x^8 + 4rstx^3 + 5rtux^4 - rs^2tx^4 + 4r^2sux^4 - 4rtu^2x^6 + r^3tux^5 - rt^3ux^7 + 4st^2ux^6 + rs^2tux^6 + 4rstu^2x^7 - 1) W_{n+1}^2,$

$\Gamma_5 = 2x^{n+5}(rs + r^2tx + tux^2 + rt^2x^2 + ru^2x^3 + r^3ux^2 - ru^3x^5 - tu^2x^4 + stx + rsux^2 - rsu^2x^4 + r^2tux^3)W_{n+4}W_{n+3},$

$\Gamma_6 = 2x^{n+5}(rt - s^2u^2x^4 + r^2ux + st^2x^2 - s^2ux^2 + t^2ux^3 - su^3x^5 + sux + rtu^2x^4 + rstux^3)W_{n+4} W_{n+2},$

$\Gamma_7 = 2ux^{n+5}(r - rux^2 + stx^2 + tux^3 - tu^2x^5 + rsux^3)W_{n+4}W_{n+1},$

$\Gamma_8 = -2x^{n+5}(-st - rt^2x + s^2tx - ru^2x^2 + ru^3x^4 + tu^2x^3 - tux - rs^2u^2x^4 + r^2tu^2x^4 + stux^2 + rst^2x^2 + rsu^2x^3 - r^2tux^2 + rt^2ux^3 - rsu^3x^5 - stu^2x^4 - rsux + r^2stux^3)W_{n+3}W_{n+2},$

$\Gamma_9 = -2ux^{n+5}(-s + s^2x - r^2ux^2 + s^2ux^3 + su^2x^4 - t^2ux^4 - rtx + rstx^2 + r^2sux^3 - rtu^2x^5) W_{n+3}W_{n+1},$

$\Gamma_{10} = -2ux^{n+5}(-t + t^3x^3 + tux^2 + rt^2x^2 + ru^2x^3 - rux + stx - stux^3 - rsu^2x^4 + r^2tux^3 + rt^2ux^4 - stu^2x^5)W_{n+2}W_{n+1}$,
$\Gamma_{11} = x^3(ux^2 + t^2x^3 + u^2x^4 - u^3x^6 + sx + rtx^2 + r^2ux^3 - su^2x^5 + rtux^4 - 1)W_3^2$,
$\Gamma_{12} = x^2(r^2x + ux^2 + t^2x^3 + u^2x^4 - u^3x^6 + sx + r^2t^2x^4 + r^2u^2x^5 - r^2u^3x^7 + rtx^2 + r^2sx^2 + r^3tx^3 + r^4ux^4 - su^2x^5 - r^2su^2x^6 + 2rstx^3 + 3rtux^4 + 2r^2sux^4 - 2rtu^2x^6 + r^3tux^5 - 1)W_2^2$,
$\Gamma_{13} = x(r^2x + ux^2 + s^2x^2 - s^3x^3 + t^2x^3 + u^2x^4 - u^3x^6 + sx + r^2t^2x^4 + r^2u^2x^5 + s^2t^2x^5 - r^2u^3x^7 - s^2u^2x^6 - s^3u^2x^7 - s^2u^3x^8 + rtx^2 + r^2sx^2 + r^3tx^3 + r^4ux^4 + s^2ux^4 - su^2x^5 - 2s^3ux^5 - r^2s^2ux^5 - r^2su^2x^6 + 4rstx^3 + 3rtux^4 - rs^2tx^4 + 4r^2sux^4 - 2rtu^2x^6 + r^3tux^5 + 2st^2ux^6 + rs^2tux^6 + 2rstu^2x^7 - 1)W_1^2$,
$\Gamma_{14} = (r^2x + ux^2 + s^2x^2 - s^3x^3 + 2t^2x^3 + u^2x^4 - t^4x^6 - u^3x^6 + sx + r^2t^2x^4 + r^2u^2x^5 + s^2t^2x^5 - r^2u^3x^7 - s^2u^2x^6 - s^3u^2x^7 - s^2u^3x^8 + t^2u^2x^7 - t^2u^3x^9 + rtx^2 + r^2sx^2 + r^3tx^3 - rt^3x^5 - st^2x^4 + r^4ux^4 + s^2ux^4 - su^2x^5 - 2s^3ux^5 - t^2ux^5 - r^2s^2ux^5 - r^2su^2x^6 - r^2t^2ux^6 + st^2u^2x^8 + 4rstx^3 + 5rtux^4 - rs^2tx^4 + 4r^2sux^4 - 4rtu^2x^6 + r^3tux^5 - rt^3ux^7 + 4st^2ux^6 + rs^2tux^6 + 4rstu^2x^7 - 1) W_0^2$,
$\Gamma_{15} = -2x^4(rs + r^2tx + tux^2 + rt^2x^2 + ru^2x^3 + r^3ux^2 - ru^3x^5 - tu^2x^4 + stx + rsux^2 - rsu^2x^4 + r^2tux^3)W_3W_2$,
$\Gamma_{16} = -2x^4(rt - s^2u^2x^4 + r^2ux + st^2x^2 - s^2ux^2 + t^2ux^3 - su^3x^5 + sux + rtu^2x^4 + rstux^3) W_3W_1$,
$\Gamma_{17} = 2x^4(-st - rt^2x + s^2tx - ru^2x^2 + ru^3x^4 + tu^2x^3 - tux - rs^2u^2x^4 + r^2tu^2x^4 + stux^2 + rst^2x^2 + rsu^2x^3 - r^2tux^2 + rt^2ux^3 - rsu^3x^5 - stu^2x^4 - rsux + r^2stux^3)W_2W_1$,
$\Gamma_{18} = -2ux^4(r - rux^2 + stx^2 + tux^3 - tu^2x^5 + rsux^3)W_3W_0$,
$\Gamma_{19} = 2ux^4(-s + s^2x - r^2ux^2 + s^2ux^3 + su^2x^4 - t^2ux^4 - rtx + rstx^2 + r^2sux^3 - rtu^2x^5)W_2W_0$,
$\Gamma_{20} = 2ux^4(-t + t^3x^3 + tux^2 + rt^2x^2 + ru^2x^3 - rux + stx - stux^3 - rsu^2x^4 + r^2tux^3 + rt^2ux^4 - stu^2x^5)W_1W_0$,
and
$\Theta_1 = x^{n+4}(r - rux^2 + stx^2 + tux^3 - tu^2x^5 + rsux^3)W_{n+4}^2$,
$\Theta_2 = x^{n+5}(rs^2 + st + rt^2x + ru^2x^2 - ru^3x^4 - tu^2x^3 + tux - r^2tu^2x^4 + r^2stx + rs^2ux^2 + r^3sux^2 + r^2tux^2 - rt^2ux^3 + rsux)W_{n+3}^2$,
$\Theta_3 = x^{n+5}(rt^2 + tu + ru^2x + st^3x^2 - ru^3x^3 - tu^2x^2 - r^2tu^2x^3 - s^2tu^2x^4 + r^2tux - rt^2ux^2 + rsu^3x^4 - s^2tux^2 + stu^2x^3 + rst^2ux^3)W_{n+2}^2$,
$\Theta_4 = u^2x^{n+5}(r - rux^2 + stx^2 + tux^3 - tu^2x^5 + rsux^3)W_{n+1}^2$,
$\Theta_5 = -x^{n+3}(r^2x + ux^2 + s^2x^2 + t^2x^3 + u^2x^4 - u^3x^6 - r^2ux^3 + s^2ux^4 - t^2ux^5 + 2rstx^3 + 2rtux^4 + 2r^2sux^4 - 2rtu^2x^6 - 1)W_{n+4}W_{n+3}$,
$\Theta_6 = x^{n+4}(t - t^3x^3 + r^2tx - s^2tx^2 + r^3ux^2 - ru^3x^5 - tu^2x^4 + rux + 2rsux^2 - rs^2ux^3 - rt^2ux^4 + 2stu^2x^5)W_{n+4}W_{n+2}$,
$\Theta_7 = ux^{n+4}(r^2x - ux^2 - s^2x^2 - t^2x^3 - u^2x^4 + u^3x^6 - r^2ux^3 - s^2ux^4 + t^2ux^5 + 1)W_{n+4}W_{n+1}$,
$\Theta_8 = -x^{n+2}(r^2x + ux^2 + s^2x^2 - s^3x^3 + t^2x^3 + u^2x^4 - u^3x^6 + sx - r^2u^3x^7 + rtx^2 - sux^3 + r^2sx^2 + r^3tx^3 - rt^3x^5 - st^2x^4 + r^4ux^4 + s^2ux^4 - su^2x^5 - s^3ux^5 - t^2ux^5 + su^3x^7 - r^2s^2ux^5 - r^2t^2ux^6 + 2rstx^3 + 2rtux^4 - rs^2tx^4 + 3r^2sux^4 - 3rtu^2x^6 + st^2ux^6 + 2rstu^2x^7 - 1)W_{n+3}W_{n+2}$,
$\Theta_9 = ux^{n+5}(t - t^3x^3 + 2rs + r^2tx + s^2tx^2 + r^3ux^2 - ru^3x^5 - tu^2x^4 + rux + 2stux^3 + rs^2ux^3 - rt^2ux^4)W_{n+3}W_{n+1}$,
$\Theta_{10} = -x^{n+1}(r^2x + ux^2 + s^2x^2 - s^3x^3 + 2t^2x^3 + u^2x^4 - t^4x^6 - u^3x^6 + sx + r^2t^2x^4 + s^2t^2x^5 - r^2u^3x^7 - s^3u^2x^7 - t^2u^2x^7 + rtx^2 + r^2sx^2 + r^3tx^3 - rt^3x^5 - st^2x^4 + r^4ux^4 + s^2ux^4 - 2su^2x^5 - 2s^3ux^5 - t^2ux^5 + su^4x^9 - r^2s^2ux^5 - r^2su^2x^6 - r^2t^2ux^6 + st^2u^2x^8 + 4rstx^3 + 3rtux^4 - rs^2tx^4 + 4r^2sux^4 - 3rtu^2x^6 + r^3tux^5 - rt^3ux^7 + 2st^2ux^6 - rtu^3x^8 + rs^2tux^6 + 2rstu^2x^7 - 1)W_{n+2}W_{n+1}$,
$\Theta_{11} = -x^3(r - rux^2 + stx^2 + tux^3 - tu^2x^5 + rsux^3)W_3^2$,
$\Theta_{12} = -x^4(rs^2 + st + rt^2x + ru^2x^2 - ru^3x^4 - tu^2x^3 + tux - r^2tu^2x^4 + r^2stx + rs^2ux^2 + r^3sux^2 + r^2tux^2 - rt^2ux^3 + rsux)W_2^2$,

$\Theta_{13} = x^4(-rt^2 - tu - ru^2x - st^3x^2 + ru^3x^3 + tu^2x^2 + r^2tu^2x^3 + s^2tu^2x^4 - r^2tux + rt^2ux^2 - rsu^3x^4 + s^2tux^2 - stu^2x^3 - rst^2ux^3)W_1^2,$

$\Theta_{14} = -u^2x^4(r - rux^2 + stx^2 + tux^3 - tu^2x^5 + rsux^3)W_0^2,$

$\Theta_{15} = x^2(r^2x + ux^2 + s^2x^2 + t^2x^3 + u^2x^4 - u^3x^6 - r^2ux^3 + s^2ux^4 - t^2ux^5 + 2rstx^3 + 2rtux^4 + 2r^2sux^4 - 2rtu^2x^6 - 1)W_3W_2,$

$\Theta_{16} = -x^3(t - t^3x^3 + r^2tx - s^2tx^2 + r^3ux^2 - ru^3x^5 - tu^2x^4 + rux + 2rsux^2 - rs^2ux^3 - rt^2ux^4 + 2stu^2x^5)W_3W_1,$

$\Theta_{17} = x(r^2x + ux^2 + s^2x^2 - s^3x^3 + t^2x^3 + u^2x^4 - u^3x^6 + sx - r^2u^3x^7 + rtx^2 - sux^3 + r^2sx^2 + r^3tx^3 - rt^3x^5 - st^2x^4 + r^4ux^4 + s^2ux^4 - su^2x^5 - s^3ux^5 - t^2ux^5 + su^3x^7 - r^2s^2ux^5 - r^2t^2ux^6 + 2rstx^3 + 2rtux^4 - rs^2tx^4 + 3r^2sux^4 - 3rtu^2x^6 + st^2ux^6 + 2rstu^2x^7 - 1)W_2W_1,$

$\Theta_{18} = ux^3(-r^2x + ux^2 + s^2x^2 + t^2x^3 + u^2x^4 - u^3x^6 + r^2ux^3 + s^2ux^4 - t^2ux^5 - 1)W_3W_0,$

$\Theta_{19} = -ux^4(t - t^3x^3 + 2rs + r^2tx + s^2tx^2 + r^3ux^2 - ru^3x^5 - tu^2x^4 + rux + 2stux^3 + rs^2ux^3 - rt^2ux^4)W_2W_0,$

$\Theta_{20} = (r^2x + ux^2 + s^2x^2 - s^3x^3 + 2t^2x^3 + u^2x^4 - t^4x^6 - u^3x^6 + sx + r^2t^2x^4 + s^2t^2x^5 - r^2u^3x^7 - s^3u^2x^7 - t^2u^2x^7 + rtx^2 + r^2sx^2 + r^3tx^3 - rt^3x^5 - st^2x^4 + r^4ux^4 + s^2ux^4 - 2su^2x^5 - 2s^3ux^5 - t^2ux^5 + su^4x^9 - r^2s^2ux^5 - r^2su^2x^6 - r^2t^2ux^6 + st^2u^2x^8 + 4rstx^3 + 3rtux^4 - rs^2tx^4 + 4r^2sux^4 - 3rtu^2x^6 + r^3tux^5 - rt^3ux^7 + 2st^2ux^6 - rtu^3x^8 + rs^2tux^6 + 2rstu^2x^7 - 1)W_1W_0,$

and

$\Phi_1 = x^{n+4}(s - s^2x + r^2 - s^2ux^3 - su^2x^4 + t^2ux^4 + rtx + rtux^3)W_{n+4}^2,$

$\Phi_2 = x^{n+3}(s - s^2x + r^2t^2x^3 + r^2u^2x^4 - r^2sx + rt^3x^4 - s^2ux^3 - su^2x^4 + t^2ux^4 - r^2s^2ux^4 - r^2su^2x^5 + r^2t^2ux^5 + rtux^3 - rs^2tx^3 + rtu^2x^5 + r^3tux^4 - 2rstux^4)W_{n+3}^2,$

$\Phi_3 = x^{n+5}(st^2 + r^2t^2 - s^2u^2x^2 - s^2u^3x^4 + rt^3x + su^2x + t^2ux + r^2u^2x - s^2t^2x - su^4x^5 + rtu + r^2t^2ux^2 + st^2u^2x^4 + r^3tux + rtu^2x^2 - rs^2tux^2 + 2rstux)W_{n+2}^2,$

$\Phi_4 = u^2x^{n+5}(s - s^2x + r^2 - s^2ux^3 - su^2x^4 + t^2ux^4 + rtx + rtux^3)W_{n+1}^2,$

$\Phi_5 = x^{n+3}(r - r^3x - t^3x^4 + tx + rs^2x^2 - r^2tx^2 - rt^2x^3 + s^2tx^3 - ru^2x^4 - tu^2x^5 + 2stux^4 + 2rs^2ux^4 + 2rsu^2x^5 - 2r^2tux^4 - 2rt^2ux^5)W_{n+4}W_{n+3},$

$\Phi_6 = -x^{n+2}(r^2x + s^2x^2 - s^3x^3 + t^2x^3 + 2u^2x^4 - u^4x^8 + sx + r^2u^2x^5 - s^2u^2x^6 + t^2u^2x^7 + sux^3 + r^2sx^2 - st^2x^4 - su^2x^5 - s^3ux^5 - su^3x^7 + 2rstx^3 + 4rtux^4 + r^2sux^4 + st^2ux^6 - 1)W_{n+4} W_{n+2},$

$\Phi_7 = ux^{n+4}(r + r^3x - t^3x^4 + tx - rs^2x^2 + r^2tx^2 - rt^2x^3 + s^2tx^3 - ru^2x^4 - tu^2x^5 + 2rsx + 2stux^4) W_{n+4}W_{n+1},$

$\Phi_8 = x^{n+3}(t - t^3x^3 + r^3u^2x^4 - r^2tx + s^2tx^2 + ru^2x^3 - s^3tx^3 + st^3x^4 - tu^2x^4 - ru^4x^7 - stx - rs^2u^2x^5 + rt^2u^2x^6 + rsux^2 + 2stux^3 + r^2stx^2 + 2rs^2ux^3 + 2rsu^2x^4 + r^3sux^3 - rs^3ux^4 + 2r^2tux^3 - 2rt^2ux^4 - rsu^3x^6 - 2s^2tux^4 + stu^2x^5 + rst^2ux^5)W_{n+3}W_{n+2},$

$\Phi_9 = -x^{n+1}(r^2x + s^2x^2 - s^3x^3 + 2t^2x^3 + 2u^2x^4 - t^4x^6 - u^4x^8 + sx + r^2t^2x^4 + r^2u^2x^5 + s^2t^2x^5 - r^2u^3x^7 - s^2u^2x^6 + rtx^2 + sux^3 + r^2sx^2 + r^3tx^3 + r^2ux^3 - rt^3x^5 - st^2x^4 + r^4ux^4 - su^2x^5 - s^3ux^5 - su^3x^7 - r^2s^2ux^5 - r^2t^2ux^6 + 4rstx^3 + 5rtux^4 - rs^2tx^4 + 3r^2sux^4 - rtu^2x^6 + r^3tux^5 - rt^3ux^7 + 3st^2ux^6 - rtu^3x^8 + rs^2tux^6 + 2rstu^2x^7 - 1)W_{n+3}W_{n+1},$

$\Phi_{10} = ux^{n+5}(2r^2t + ru + 2st + 2rt^2x - 2s^2tx + r^3ux - ru^3x^4 + t^3ux^4 - tu^3x^5 + tux - rs^2ux^2 + r^2tux^2 + rt^2ux^3 - s^2tux^3 + 2rsux)W_{n+2}W_{n+1},$

$\Phi_{11} = -x^3(s - s^2x + r^2 - s^2ux^3 - su^2x^4 + t^2ux^4 + rtx + rtux^3)W_3^2,$

$\Phi_{12} = -x^2(s - s^2x + r^2t^2x^3 + r^2u^2x^4 - r^2sx + rt^3x^4 - s^2ux^3 - su^2x^4 + t^2ux^4 - r^2s^2ux^4 - r^2su^2x^5 + r^2t^2ux^5 + rtux^3 - rs^2tx^3 + rtu^2x^5 + r^3tux^4 - 2rstux^4)W_2^2,$

$\Phi_{13} = -x^4(st^2 + r^2t^2 - s^2u^2x^2 - s^2u^3x^4 + rt^3x + su^2x + t^2ux + r^2u^2x - s^2t^2x - su^4x^5 + rtu + r^2t^2ux^2 + st^2u^2x^4 + r^3tux + rtu^2x^2 - rs^2tux^2 + 2rstux)W_1^2,$

$\Phi_{14} = -u^2x^4(s - s^2x + r^2 - s^2ux^3 - su^2x^4 + t^2ux^4 + rtx + rtux^3)W_0^2,$

$\Phi_{15} = x^2(-r + r^3x + t^3x^4 - tx - rs^2x^2 + r^2tx^2 + rt^2x^3 - s^2tx^3 + ru^2x^4 + tu^2x^5 - 2stux^4 - 2rs^2ux^4 - 2rsu^2x^5 + 2r^2tux^4 + 2rt^2ux^5)W_3W_2,$

$\Phi_{16} = x(r^2x + s^2x^2 - s^3x^3 + t^2x^3 + 2u^2x^4 - u^4x^8 + sx + r^2u^2x^5 - s^2u^2x^6 + t^2u^2x^7 + sux^3 + r^2sx^2 - st^2x^4 - su^2x^5 - s^3ux^5 - su^3x^7 + 2rstx^3 + 4rtux^4 + r^2sux^4 + st^2ux^6 - 1)W_3W_1,$

$\Phi_{17} = -x^2(t - t^3x^3 + r^3u^2x^4 - r^2tx + s^2tx^2 + ru^2x^3 - s^3tx^3 + st^3x^4 - tu^2x^4 - ru^4x^7 - stx - rs^2u^2x^5 + rt^2u^2x^6 + rsux^2 + 2stux^3 + r^2stx^2 + 2rs^2ux^3 + 2rsu^2x^4 + r^3sux^3 - rs^3ux^4 + 2r^2tux^3 - 2rt^2ux^4 - rsu^3x^6 - 2s^2tux^4 + stu^2x^5 + rst^2ux^5)W_2W_1,$

$\Phi_{18} = -ux^3(r + r^3x - t^3x^4 + tx - rs^2x^2 + r^2tx^2 - rt^2x^3 + s^2tx^3 - ru^2x^4 - tu^2x^5 + 2rsx + 2stux^4) W_3W_0,$

$\Phi_{19} = (r^2x + s^2x^2 - s^3x^3 + 2t^2x^3 + 2u^2x^4 - t^4x^6 - u^4x^8 + sx + r^2t^2x^4 + r^2u^2x^5 + s^2t^2x^5 - r^2u^3x^7 - s^2u^2x^6 + rtx^2 + sux^3 + r^2sx^2 + r^3tx^3 + r^2ux^3 - rt^3x^5 - st^2x^4 + r^4ux^4 - su^2x^5 - s^3ux^5 - su^3x^7 - r^2s^2ux^5 - r^2t^2ux^6 + 4rstx^3 + 5rtux^4 - rs^2tx^4 + 3r^2sux^4 - rtu^2x^6 + r^3tux^5 - rt^3ux^7 + 3st^2ux^6 - rtu^3x^8 + rs^2tux^6 + 2rstu^2x^7 - 1)W_2W_0,$

$\Phi_{20} = -ux^4(2r^2t + ru + 2st + 2rt^2x - 2s^2tx + r^3ux - ru^3x^4 + t^3ux^4 - tu^3x^5 + tux - rs^2ux^2 + r^2tux^2 + rt^2ux^3 - s^2tux^3 + 2rsux)W_1W_0,$

and

$\Psi_1 = x^{n+4}(t - t^3x^3 + 2rs + r^3 - rs^2x + r^2tx - tux^2 - rt^2x^2 + s^2tx^2 - ru^2x^3 + rux - stx - rsux^2 + 2stux^3)W_{n+4}^2,$

$\Psi_2 = x^{n+3}(t - t^3x^3 + rs + r^3u^2x^4 - rs^2x - r^3sx - r^2tx - tux^2 + rs^3x^2 - rt^2x^2 + s^2tx^2 + ru^3x^5 - ru^4x^7 - stx + rs^2u^2x^5 + r^2tu^2x^5 + rt^2u^2x^6 + 2stux^3 - r^2stx^2 - rst^2x^3 + rs^2ux^3 + rsu^2x^4 + rs^3ux^4 + r^2tux^3 - rt^2ux^4 - rsu^3x^6 - 2r^2stux^4 - rst^2ux^5)W_{n+3}^2,$

$\Psi_3 = x^{n+2}(t - t^3x^3 + r^3u^2x^4 - r^2tx - tux^2 - rt^2x^2 - s^2tx^2 + s^3tx^3 + ru^3x^5 - ru^4x^7 - stx - rs^2u^2x^5 + r^2tu^2x^5 + rt^2u^2x^6 + s^2tu^2x^6 + stux^3 - r^2stx^2 - 2rst^2x^3 + 2rsu^2x^4 + r^2tux^3 - rt^2ux^4 - rsu^3x^6 - s^2tux^4 + s^3tux^5 - st^3ux^6 + stu^3x^7 - r^2stux^4 - rst^2ux^5)W_{n+2}^2,$

$\Psi_4 = u^2x^{n+5}(t - t^3x^3 + 2rs + r^3 - rs^2x + r^2tx - tux^2 - rt^2x^2 + s^2tx^2 - ru^2x^3 + rux - stx - rsux^2 + 2stux^3)W_{n+1}^2,$

$\Psi_5 = x^{n+3}(s - r^4x - s^3x^2 - u^2x^3 - u^3x^5 + u^4x^7 + ux + r^2 + r^2s^2x^2 + r^2t^2x^3 - s^2u^2x^5 - t^2u^2x^6 - r^2sx - sux^2 - r^3tx^2 - r^2ux^2 + rt^3x^4 + st^2x^3 - s^2ux^3 - su^2x^4 - s^3ux^4 + t^2ux^4 + su^3x^6 + rtx + 2rstx^2 - rs^2tx^3 + r^2sux^3 - rtu^2x^5 + st^2ux^5)W_{n+4}W_{n+3},$

$\Psi_6 = x^{n+2}(r - r^3x + stx^2 + tux^3 - rs^2x^2 - r^3sx^2 + rs^3x^3 + rt^2x^3 - ru^2x^4 - s^3tx^4 + st^3x^5 + t^3ux^6 - tu^3x^7 - rsx + 2stux^4 - r^2stx^3 + rst^2x^4 + 2rs^2ux^4 + 3rsu^2x^5 - r^2tux^4 - 3s^2tux^5 - stu^2x^6) W_{n+4}W_{n+2},$

$\Psi_7 = -x^{n+1}(r^2x + ux^2 + s^2x^2 - s^3x^3 + 2t^2x^3 + u^2x^4 - t^4x^6 - u^3x^6 + sx + r^2t^2x^4 + s^2t^2x^5 - t^2u^2x^7 + rtx^2 - sux^3 + r^2sx^2 + r^3tx^3 - r^2ux^3 - rt^3x^5 - st^2x^4 + s^2ux^4 - su^2x^5 - s^3ux^5 - t^2ux^5 + su^3x^7 + 4rstx^3 + 2rtux^4 - rs^2tx^4 + r^2sux^4 - 3rtu^2x^6 + 3st^2ux^6 - 1)W_{n+4}W_{n+1},$

$\Psi_8 = x^{n+2}(s - s^2x - s^3x^2 + s^4x^3 - u^2x^3 - u^3x^5 + u^4x^7 + ux - r^2s^2x^2 + r^2u^2x^4 - s^2t^2x^4 + s^3u^2x^6 - s^2u^3x^7 - t^2u^2x^6 - r^2sx - 2sux^2 - r^3tx^2 - r^2ux^2 + rt^3x^4 + st^2x^3 + t^2ux^4 + 2su^3x^6 + s^4ux^5 - su^4x^8 + rtx - r^2s^2ux^4 - r^2su^2x^5 - s^2t^2ux^6 + st^2u^2x^7 - 2rstx^2 - rtux^3 - rs^2tx^3 - rtu^2x^5 + r^3tux^4 - rt^3ux^6 + rtu^3x^7 - rs^2tux^5 + 2rstu^2x^6)W_{n+3}W_{n+2},$

$\Psi_9 = ux^{n+3}(r - r^3x + 3stx^2 + tux^3 + 3rs^2x^2 + r^3sx^2 - rs^3x^3 + rt^2x^3 - 2s^2tx^3 - ru^2x^4 + s^3tx^4 - st^3x^5 + t^3ux^6 - tu^3x^7 - rsx + 2rsux^3 + r^2stx^3 - rst^2x^4 + rsu^2x^5 - r^2tux^4 + s^2tux^5 - stu^2x^6) W_{n+3}W_{n+1},$

$\Psi_{10} = ux^{n+2}(-r^2x - ux^2 - s^2x^2 + s^3x^3 - u^2x^4 - t^4x^6 + u^3x^6 - sx + r^2t^2x^4 + s^2t^2x^5 + t^2u^2x^7 - rtx^2 + sux^3 - r^2sx^2 + r^3tx^3 + r^2ux^3 - rt^3x^5 - st^2x^4 - s^2ux^4 + su^2x^5 + s^3ux^5 - t^2ux^5 - su^3x^7 - rs^2tx^4 - r^2sux^4 + rtu^2x^6 + st^2ux^6 - 2rstux^5 + 1)W_{n+2}W_{n+1},$

$\Psi_{11} = -x^3(t - t^3x^3 + 2rs + r^3 - rs^2x + r^2tx - tux^2 - rt^2x^2 + s^2tx^2 - ru^2x^3 + rux - stx - rsux^2 + 2stux^3)W_3^2,$

$\Psi_{12} = x^2(-t + t^3x^3 - rs - r^3u^2x^4 + rs^2x + r^3sx + r^2tx + tux^2 - rs^3x^2 + rt^2x^2 - s^2tx^2 - ru^3x^5 + ru^4x^7 + stx - rs^2u^2x^5 - r^2tu^2x^5 - rt^2u^2x^6 - 2stux^3 + r^2stx^2 + rst^2x^3 - rs^2ux^3 - rsu^2x^4 - rs^3ux^4 - r^2tux^3 + rt^2ux^4 + rsu^3x^6 + 2r^2stux^4 + rst^2ux^5)W_2^2,$

$\Psi_{13} = -x(t - t^3x^3 + r^3u^2x^4 - r^2tx - tux^2 - rt^2x^2 - s^2tx^2 + s^3tx^3 + ru^3x^5 - ru^4x^7 - stx - rs^2u^2x^5 + r^2tu^2x^5 + rt^2u^2x^6 + s^2tu^2x^6 + stux^3 - r^2stx^2 - 2rst^2x^3 + 2rsu^2x^4 + r^2tux^3 - rt^2ux^4 - rsu^3x^6 - s^2tux^4 + s^3tux^5 - st^3ux^6 + stu^3x^7 - r^2stux^4 - rst^2ux^5)W_1^2,$

$\Psi_{14} = -u^2x^4(t - t^3x^3 + 2rs + r^3 - rs^2x + r^2tx - tux^2 - rt^2x^2 + s^2tx^2 - ru^2x^3 + rux - stx - rsux^2 + 2stux^3)W_0^2,$

$\Psi_{15} = x^2(-s + r^4x + s^3x^2 + u^2x^3 + u^3x^5 - u^4x^7 - ux - r^2 - r^2s^2x^2 - r^2t^2x^3 + s^2u^2x^5 + t^2u^2x^6 + r^2sx + sux^2 + r^3tx^2 + r^2ux^2 - rt^3x^4 - st^2x^3 + s^2ux^3 + su^2x^4 + s^3ux^4 - t^2ux^4 - su^3x^6 - rtx - 2rstx^2 + rs^2tx^3 - r^2sux^3 + rtu^2x^5 - st^2ux^5)W_3W_2,$

$\Psi_{16} = x(-r + r^3x - stx^2 - tux^3 + rs^2x^2 + r^3sx^2 - rs^3x^3 - rt^2x^3 + ru^2x^4 + s^3tx^4 - st^3x^5 - t^3ux^6 + tu^3x^7 + rsx - 2stux^4 + r^2stx^3 - rst^2x^4 - 2rs^2ux^4 - 3rsu^2x^5 + r^2tux^4 + 3s^2tux^5 + stu^2x^6) W_3W_1,$

$\Psi_{17} = -x(s - s^2x - s^3x^2 + s^4x^3 - u^2x^3 - u^3x^5 + u^4x^7 + ux - r^2s^2x^2 + r^2u^2x^4 - s^2t^2x^4 + s^3u^2x^6 - s^2u^3x^7 - t^2u^2x^6 - r^2sx - 2sux^2 - r^3tx^2 - r^2ux^2 + rt^3x^4 + st^2x^3 + t^2ux^4 + 2su^3x^6 + s^4ux^5 - su^4x^8 + rtx - r^2s^2ux^4 - r^2su^2x^5 - s^2t^2ux^6 + st^2u^2x^7 - 2rstx^2 - rtux^3 - rs^2tx^3 - rtu^2x^5 + r^3tux^4 - rt^3ux^6 + rtu^3x^7 - rs^2tux^5 + 2rstu^2x^6)W_2W_1,$

$\Psi_{18} = (r^2x + ux^2 + s^2x^2 - s^3x^3 + 2t^2x^3 + u^2x^4 - t^4x^6 - u^3x^6 + sx + r^2t^2x^4 + s^2t^2x^5 - t^2u^2x^7 + rtx^2 - sux^3 + r^2sx^2 + r^3tx^3 - r^2ux^3 - rt^3x^5 - st^2x^4 + s^2ux^4 - su^2x^5 - s^3ux^5 - t^2ux^5 + su^3x^7 + 4rstx^3 + 2rtux^4 - rs^2tx^4 + r^2sux^4 - 3rtu^2x^6 + 3st^2ux^6 - 1)W_3W_0,$

$\Psi_{19} = -ux^2(r - r^3x + 3stx^2 + tux^3 + 3rs^2x^2 + r^3sx^2 - rs^3x^3 + rt^2x^3 - 2s^2tx^3 - ru^2x^4 + s^3tx^4 - st^3x^5 + t^3ux^6 - tu^3x^7 - rsx + 2rsux^3 + r^2stx^3 - rst^2x^4 + rsu^2x^5 - r^2tux^4 + s^2tux^5 - stu^2x^6) W_2W_0,$

$\Psi_{20} = ux(r^2x + ux^2 + s^2x^2 - s^3x^3 + u^2x^4 + t^4x^6 - u^3x^6 + sx - r^2t^2x^4 - s^2t^2x^5 - t^2u^2x^7 + rtx^2 - sux^3 + r^2sx^2 - r^3tx^3 - r^2ux^3 + rt^3x^5 + st^2x^4 + s^2ux^4 - su^2x^5 - s^3ux^5 + t^2ux^5 + su^3x^7 + rs^2tx^4 + r^2sux^4 - rtu^2x^6 - st^2ux^6 + 2rstux^5 - 1)W_1W_0.$

Proof. First, we obtain $\sum_{k=0}^{n} x^kW_k^2$. Using the recurrence relation

$$W_{n+4} = rW_{n+3} + sW_{n+2} + tW_{n+1} + uW_n$$

or

$$uW_n = W_{n+4} - rW_{n+3} - sW_{n+2} - tW_{n+1}$$

i.e.

$$\begin{aligned} u^2W_n^2 &= W_{n+4}^2 + r^2W_{n+3}^2 + s^2W_{n+2}^2 + t^2W_{n+1}^2 - 2rW_{n+4}W_{n+3} - 2sW_{n+4}W_{n+2} \\ &\quad -2tW_{n+4}W_{n+1} + 2rsW_{n+3}W_{n+2} + 2rtW_{n+3}W_{n+1} + 2stW_{n+2}W_{n+1} \end{aligned}$$

we obtain

$$\begin{aligned} u^2x^nW_n^2 &= x^nW_{n+4}^2 + r^2x^nW_{n+3}^2 + s^2x^nW_{n+2}^2 + t^2x^nW_{n+1}^2 - 2rx^nW_{n+4}W_{n+3} - 2sx^nW_{n+4}W_{n+2} \\ &\quad -2tx^nW_{n+4}W_{n+1} + 2rsx^nW_{n+3}W_{n+2} + 2rtx^nW_{n+3}W_{n+1} + 2stx^nW_{n+2}W_{n+1} \\ u^2x^{n-1}W_{n-1}^2 &= x^{n-1}W_{n+3}^2 + r^2x^{n-1}W_{n+2}^2 + s^2x^{n-1}W_{n+1}^2 + t^2x^{n-1}W_n^2 - 2rx^{n-1}W_{n+3}W_{n+2} - 2sx^{n-1}W_{n+3}W_{n+1} \\ &\quad -2tx^{n-1}W_{n+3}W_n + 2rsx^{n-1}W_{n+2}W_{n+1} + 2rtx^{n-1}W_{n+2}W_n + 2stx^{n-1}W_{n+1}W_n \\ u^2x^{n-2}W_{n-2}^2 &= x^{n-2}W_{n+2}^2 + r^2x^{n-2}W_{n+1}^2 + s^2x^{n-2}W_n^2 + t^2x^{n-2}W_{n-1}^2 - 2rx^{n-2}W_{n+2}W_{n+1} - 2sx^{n-2}W_{n+2}W_n \\ &\quad -2tx^{n-2}W_{n+2}W_{n-1} + 2rsx^{n-2}W_{n+1}W_n + 2rtx^{n-2}W_{n+1}W_{n-1} + 2stx^{n-2}W_nW_{n-1} \\ &\vdots \\ u^2x^2W_2^2 &= x^2W_6^2 + r^2x^2W_5^2 + s^2x^2W_4^2 + t^2x^2W_3^2 - 2rx^2W_6W_5 - 2sx^2W_6W_4 \\ &\quad -2tx^2W_6W_3 + 2rsx^2W_5W_4 + 2rtx^2W_5W_3 + 2stx^2W_4W_3 \\ u^2x^1W_1^2 &= x^1W_5^2 + r^2x^1W_4^2 + s^2x^1W_3^2 + t^2x^1W_2^2 - 2rx^1W_5W_4 - 2sx^1W_5W_3 \\ &\quad -2tx^1W_5W_2 + 2rsx^1W_4W_3 + 2rtx^1W_4W_2 + 2stx^1W_3W_2 \\ u^2x^0W_0^2 &= x^0W_4^2 + r^2x^0W_3^2 + s^2x^0W_2^2 + t^2x^0W_1^2 - 2rx^0W_4W_3 - 2sx^0W_4W_2 \\ &\quad -2tx^0W_4W_1 + 2rsx^0W_3W_2 + 2rtx^0W_3W_1 + 2stx^0W_2W_1 \end{aligned}$$

If we add the equations side by side, we get

$$
\begin{aligned}
u^2\sum_{k=0}^{n}x^kW_k^2 &= (r^2x+s^2x^2+t^2x^3+1)x^{-4}\sum_{k=0}^{n}x^kW_k^2-2tx^{-1}\sum_{k=0}^{n}x^kW_{k+3}W_k \qquad (2)\\
&+2(-s+rtx)x^{-2}\sum_{k=0}^{n}x^kW_{k+2}W_k+2(-r+stx^2+rsx)x^{-3}\sum_{k=0}^{n}x^kW_{k+1}W_k\\
&+x^nW_{n+4}^2+(r^2x+1)x^{n-1}W_{n+3}^2+(r^2x+s^2x^2+1)x^{n-2}W_{n+2}^2\\
&+(r^2x+s^2x^2+t^2x^3+1)x^{n-3}W_{n+1}^2-2rx^nW_{n+4}W_{n+3}-2sx^nW_{n+4}W_{n+2}\\
&-2tx^nW_{n+4}W_{n+1}+2r(sx-1)x^{n-1}W_{n+3}W_{n+2}+2(-s+rtx)x^{n-1}W_{n+3}W_{n+1}\\
&+2(-r+stx^2+rsx)x^{n-2}W_{n+2}W_{n+1}-x^{-1}W_3^2-(r^2x+1)x^{-2}W_2^2\\
&-(r^2x+s^2x^2+1)x^{-3}W_1^2-(r^2x+s^2x^2+t^2x^3+1)x^{-4}W_0^2\\
&+2rx^{-1}W_3W_2+2sx^{-1}W_3W_1-2r(sx-1)x^{-2}W_2W_1+2tx^{-1}W_3W_0\\
&-2(-s+rtx)x^{-2}W_2W_0-2(-r+stx^2+rsx)x^{-3}W_1W_0.
\end{aligned}
$$

Next we obtain $\sum_{k=0}^{n}x^kW_{k+1}W_k$. Multiplying the both side of the recurrence relation

$$uW_n=W_{n+4}-rW_{n+3}-sW_{n+2}-tW_{n+1}$$

by W_{n+1} we get

$$uW_{n+1}W_n=W_{n+4}W_{n+1}-rW_{n+3}W_{n+1}-sW_{n+2}W_{n+1}-tW_{n+1}^2$$

Then using last recurrence relation, we obtain

$$
\begin{aligned}
ux^nW_{n+1}W_n &= x^nW_{n+4}W_{n+1}-rx^nW_{n+3}W_{n+1}-sx^nW_{n+2}W_{n+1}-tx^nW_{n+1}^2\\
ux^{n-1}W_nW_{n-1} &= x^{n-1}W_{n+3}W_n-rx^{n-1}W_{n+2}W_n-sx^{n-1}W_{n+1}W_n-tx^{n-1}W_n^2\\
ux^{n-2}W_{n-1}W_{n-2} &= x^{n-2}W_{n+2}W_{n-1}-rx^{n-2}W_{n+1}W_{n-1}-sx^{n-2}W_nW_{n-1}-tx^{n-2}W_{n-1}^2\\
&\vdots\\
ux^2W_3W_2 &= x^2W_6W_3-rx^2W_5W_3-sx^2W_4W_3-tx^2W_3^2\\
ux^1W_2W_1 &= x^1W_5W_2-rx^1W_4W_2-sx^1W_3W_2-tx^1W_2^2\\
ux^0W_1W_0 &= x^0W_4W_1-rx^0W_3W_1-sx^0W_2W_1-tx^0W_1^2
\end{aligned}
$$

If we add the equations side by side, we get

$$
\begin{aligned}
u\sum_{k=0}^{n}x^kW_{k+1}W_k &= (x^nW_{n+4}W_{n+1}-x^{-1}W_3W_0+x^{-1}\sum_{k=0}^{n}x^kW_{k+3}W_k) \qquad (3)\\
&-r(x^nW_{n+3}W_{n+1}-x^{-1}W_2W_0+x^{-1}\sum_{k=0}^{n}x^kW_{k+2}W_k)\\
&-s(x^nW_{n+2}W_{n+1}-x^{-1}W_1W_0+x^{-1}\sum_{k=0}^{n}x^kW_{k+1}W_k)\\
&-t(x^nW_{n+1}^2-x^{-1}W_0^2+x^{-1}\sum_{k=0}^{n}x^kW_k^2)
\end{aligned}
$$

Next we obtain $\sum_{k=0}^{n}x^kW_{k+2}W_k$. Multiplying the both side of the recurrence relation

$$uW_n=W_{n+4}-rW_{n+3}-sW_{n+2}-tW_{n+1}$$

by W_{n+2} we get

$$uW_{n+2}W_n = W_{n+4}W_{n+2} - rW_{n+3}W_{n+2} - sW_{n+2}^2 - tW_{n+2}W_{n+1}$$

Then using last recurrence relation, we obtain

$$\begin{array}{rcl} ux^nW_{n+2}W_n & = & x^nW_{n+4}W_{n+2} - rx^nW_{n+3}W_{n+2} - sx^nW_{n+2}^2 - tx^nW_{n+2}W_{n+1} \\ ux^{n-1}W_{n+1}W_{n-1} & = & x^{n-1}W_{n+3}W_{n+1} - rx^{n-1}W_{n+2}W_{n+1} - sx^{n-1}W_{n+1}^2 - tx^{n-1}W_{n+1}W_n \\ ux^{n-2}W_nW_{n-2} & = & x^{n-2}W_{n+2}W_n - rx^{n-2}W_{n+1}W_n - sx^{n-2}W_n^2 - tx^{n-2}W_nW_{n-1} \\ & \vdots & \\ ux^2W_4W_2 & = & x^2W_6W_4 - rx^2W_5W_4 - sx^2W_4^2 - tx^2W_4W_3 \\ ux^1W_3W_1 & = & x^1W_5W_3 - rx^1W_4W_3 - sx^1W_3^2 - tx^1W_3W_2 \\ ux^0W_2W_0 & = & x^0W_4W_2 - rx^0W_3W_2 - sx^0W_2^2 - tx^0W_2W_1 \end{array}$$

If we add the equations side by side, we get

$$\begin{array}{rcl} u\sum_{k=0}^{n} x^kW_{k+2}W_k & = & (x^nW_{n+4}W_{n+2} + x^{n-1}W_{n+3}W_{n+1} - x^{-1}W_3W_1 - x^{-2}W_2W_0 + x^{-2}\sum_{k=0}^{n} x^kW_{k+2}W_k) \quad (4) \\ & & -r(x^nW_{n+3}W_{n+2} + x^{n-1}W_{n+2}W_{n+1} - x^{-1}W_2W_1 - x^{-2}W_1W_0 + x^{-2}\sum_{k=0}^{n} x^kW_{k+1}W_k) \\ & & -s(x^nW_{n+2}^2 + x^{n-1}W_{n+1}^2 - x^{-1}W_1^2 - x^{-2}W_0^2 + x^{-2}\sum_{k=0}^{n} x^kW_k^2) \\ & & -t(x^nW_{n+2}W_{n+1} - x^{-1}W_1W_0 + x^{-1}\sum_{k=0}^{n} x^kW_{k+1}W_k) \end{array}$$

Next we obtain $\sum_{k=0}^{n} x^kW_{k+3}W_k$. Multiplying the both side of the recurrence relation

$$uW_n = W_{n+4} - rW_{n+3} - sW_{n+2} - tW_{n+1}$$

by W_{n+3} we get

$$uW_{n+3}W_n = W_{n+4}W_{n+3} - rW_{n+3}^2 - sW_{n+3}W_{n+2} - tW_{n+3}W_{n+1}$$

Then using last recurrence relation, we obtain

$$\begin{array}{rcl} ux^nW_{n+3}W_n & = & x^nW_{n+4}W_{n+3} - rx^nW_{n+3}^2 - sx^nW_{n+3}W_{n+2} - tx^nW_{n+3}W_{n+1} \\ ux^{n-1}W_{n+2}W_{n-1} & = & x^{n-1}W_{n+3}W_{n+2} - rx^{n-1}W_{n+2}^2 - sx^{n-1}W_{n+2}W_{n+1} - tx^{n-1}W_{n+2}W_n \\ ux^{n-2}W_{n+1}W_{n-2} & = & x^{n-2}W_{n+2}W_{n+1} - rx^{n-2}W_{n+1}^2 - sx^{n-2}W_{n+1}W_n - tx^{n-2}W_{n+1}W_{n-1} \\ & \vdots & \\ ux^2W_5W_2 & = & x^2W_6W_5 - rx^2W_5^2 - sx^2W_5W_4 - tx^2W_5W_3 \\ ux^1W_4W_1 & = & x^1W_5W_4 - rx^1W_4^2 - sx^1W_4W_3 - tx^1W_4W_2 \\ ux^0W_3W_0 & = & x^0W_4W_3 - rx^0W_3^2 - sx^0W_3W_2 - tx^0W_3W_1 \end{array}$$

If we add the equations side by side, we get

$$\begin{array}{rcl} u\sum_{k=0}^{n} x^kW_{k+3}W_k & = & (x^nW_{n+4}W_{n+3} + x^{n-1}W_{n+3}W_{n+2} + x^{n-2}W_{n+2}W_{n+1} \quad (5) \\ & & -x^{-1}W_3W_2 - x^{-2}W_2W_1 - x^{-3}W_1W_0 + x^{-3}\sum_{k=0}^{n} x^kW_{k+1}W_k) \\ & & -r(x^nW_{n+3}^2 + x^{n-1}W_{n+2}^2 + x^{n-2}W_{n+1}^2 - x^{-1}W_2^2 - x^{-2}W_1^2 - x^{-3}W_0^2 + x^{-3}\sum_{k=0}^{n} x^kW_k^2) \\ & & -s(x^nW_{n+3}W_{n+2} + x^{n-1}W_{n+2}W_{n+1} - x^{-1}W_2W_1 - x^{-2}W_1W_0 + x^{-2}\sum_{k=0}^{n} x^kW_{k+1}W_k) \\ & & -t(x^nW_{n+3}W_{n+1} - x^{-1}W_2W_0 + x^{-1}\sum_{k=0}^{n} x^kW_{k+2}W_k) \end{array}$$

Solving the system (2)-(3)-(4)-(5), the results in (a), (b), (c) and (d) follow.

4 Specific Cases

In this section, for the specific cases of x, we present the closed form solutions (identities) of the sums $\sum_{k=0}^{n} x^k W_k^2$, $\sum_{k=0}^{n} x^k W_{k+1}W_k$, $\sum_{k=0}^{n} x^k W_{k+2}W_k$ and $\sum_{k=0}^{n} x^k W_{k+3}W_k$ for the specific case of sequence $\{W_n\}$.

4.1 The case $x = 1$

In this subsection we consider the special case $x = 1$.

The case $x = 1$ of Theorem 3.1 is given in [29].

We only consider the case $x = 1, r = 1, s = 1, t = 1, u = 2$ (which is not considered in [29]). Observe that setting $x = 1, r = 1, s = 1, t = 1, u = 2$ (i.e. for the generalized fourth order Jacobsthal case) in Theorem 3.1 (a),(b),(c) and (d) makes the right hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule (using twice) however provides the evaluation of the sum formulas.

Taking $r = 1, s = 1, t = 1, u = 2$ in Theorem 3.1, we obtain the following theorem.

Theorem 4.1. *If $r = 1, s = 1, t = 1, u = 2$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} W_k^2 = \frac{1}{180}(14(n+8)W_{n+4}^2 + (38n+279)W_{n+3}^2 + 2(25n+164)W_{n+2}^2 + 4(14n+81)W_{n+1}^2 - (38n+293)W_{n+4}W_{n+3} - (26n+199)W_{n+4}W_{n+2} - 2(10n+79)W_{n+4}W_{n+1} - (2n-13)W_{n+3}W_{n+2} + 2(2n+27)W_{n+3}W_{n+1} + 4(4n+37)W_{n+2}W_{n+1} - 98W_3^2 - 241W_2^2 - 278W_1^2 - 268W_0^2 + 255W_3W_2 + 173W_3W_1 + 138W_3W_0 - 15W_2W_1 - 50W_2W_0 - 132W_1W_0)$

(b) $\sum_{k=0}^{n} W_{k+1}W_k = \frac{1}{360}(-(10n+69)W_{n+4}^2 - 40(n+7)W_{n+3}^2 - (10n+51)W_{n+2}^2 - 4(10n+79)W_{n+1}^2 + 10(4n+29)W_{n+4}W_{n+3} - 20(n+6)W_{n+4}W_{n+2} + 8(5n+43)W_{n+4}W_{n+1} + 2(20n+109)W_{n+3}W_{n+2} - 4(20n+149)W_{n+3}W_{n+1} + 4(10n+23)W_{n+2}W_{n+1} + 59W_3^2 + 240W_2^2 + 41W_1^2 + 276W_0^2 - 250W_3W_2 + 100W_3W_1 - 304W_3W_0 - 178W_2W_1 + 516W_2W_0 - 52W_1W_0)$

(c) $\sum_{k=0}^{n} W_{k+2}W_k = \frac{1}{180}(-2(2n+11)W_{n+4}^2 + (2n+13)W_{n+3}^2 - 4(10n+73)W_{n+2}^2 - 8(2n+13)W_{n+1}^2 - (2n+9)W_{n+4}W_{n+3} + (46n+325)W_{n+4}W_{n+2} - 2(10n+63)W_{n+4}W_{n+1} - (38n+279)W_{n+3}W_{n+2} + 2(38n+229)W_{n+3}W_{n+1} - 4(14n+99)W_{n+2}W_{n+1} + 18W_3^2 - 11W_2^2 + 252W_1^2 + 88W_0^2 + 7W_3W_2 - 279W_3W_1 + 106W_3W_0 + 241W_2W_1 - 382W_2W_0 + 340W_1W_0)$

(d) $\sum_{k=0}^{n} W_{k+3}W_k = \frac{1}{360}(-(10n+47)W_{n+4}^2 - 8(5n+42)W_{n+3}^2 - (10n+101)W_{n+2}^2 - 4(10n+57)W_{n+1}^2 + 2(20n+173)W_{n+4}W_{n+3} - 4(5n+37)W_{n+4}W_{n+2} + 8(5n+23)W_{n+4}W_{n+1} + 2(20n+101)W_{n+3}W_{n+2} - 4(20n+141)W_{n+3}W_{n+1} + 20(2n+11)W_{n+2}W_{n+1} + 37W_3^2 + 296W_2^2 + 91W_1^2 + 188W_0^2 - 306W_3W_2 + 128W_3W_1 - 144W_3W_0 - 162W_2W_1 + 484W_2W_0 - 180W_1W_0)$

Proof.

(a) We use Theorem 3.1 (a). If we set $r = 1, s = 1, t = 1, u = 2$ in Theorem 3.1 (a) then we have

$$\sum_{k=0}^{n} x^k W_k^2 = \frac{g_1(x)}{(4x-1)(x-1)^2(x+1)^2(2x+1)(4x^2+1)(x^2+1)}$$

where
$g_1(x) = -x^{n+4}(-8x^6 - 4x^5 + 6x^4 + 3x^3 + 3x^2 + x - 1)W_{n+4}^2 - x^{n+3}(-8x^7 - 20x^6 + 2x^5 + 17x^4 + 4x^3 + 4x^2 + 2x - 1)W_{n+3}^2 + x^{n+2}(8x^8 + 4x^7 + 18x^6 + 3x^5 - 22x^4 - 5x^3 - 5x^2 - 2x + 1)W_{n+2}^2 - x^{n+1}(-8x^9 - 4x^8 + 6x^7 - 25x^6 - 6x^5 + 25x^4 + 6x^3 + 5x^2 + 2x - 1)W_{n+1}^2 + 2x^{n+5}(-8x^5 - 8x^4 + 6x^3 + 7x^2 + 2x + 1)W_{n+4}W_{n+3} + 2x^{n+5}(-8x^5 + 4x^3 - x^2 + 4x + 1)W_{n+2}W_{n+4} +$

$4x^{n+5}(-4x^5+4x^3-x^2+1)W_{n+4}W_{n+1}+2x^{n+5}(8x^5-4x^4-12x^3+3x^2+4x+1)W_{n+2}W_{n+3}+4x^{n+5}(4x^5-2x^4-4x^3+x^2+1)W_{n+3}W_{n+1}+4x^{n+5}(4x^5+2x^4-5x^3-3x^2+x+1)W_{n+2}W_{n+1}+x^3(-8x^6-4x^5+6x^4+3x^3+3x^2+x-1)W_3^2+x^2(-8x^7-20x^6+2x^5+17x^4+4x^3+4x^2+2x-1)W_2^2-x(8x^8+4x^7+18x^6+3x^5-22x^4-5x^3-5x^2-2x+1)W_1^2+(-8x^9-4x^8+6x^7-25x^6-6x^5+25x^4+6x^3+5x^2+2x-1)W_0^2-2x^4(-8x^5-8x^4+6x^3+7x^2+2x+1)W_3W_2-2x^4(8x^5-4x^4-12x^3+3x^2+4x+1)W_2W_1-2x^4(-8x^5+4x^3-x^2+4x+1)W_3W_1-4x^4(-4x^5+4x^3-x^2+1)W_3W_0-4x^4(4x^5-2x^4-4x^3+x^2+1)W_2W_0-4x^4(4x^5+2x^4-5x^3-3x^2+x+1)W_1W_0$

For $x=1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (a) using

$$\sum_{k=0}^{n} W_k^2 = \left.\frac{\frac{d^2}{dx^2}(g_1(x))}{\frac{d^2}{dx^2}((4x-1)(x-1)^2(x+1)^2(2x+1)(4x^2+1)(x^2+1))}\right|_{x=1}$$

(b) We use Theorem 3.1 (b). If we set $r=1,s=1,t=1,u=2$ in Theorem 3.1 (b) then we have

$$\sum_{k=0}^{n} x^k W_{k+1}W_k = \frac{g_2(x)}{(4x-1)(x-1)^2(x+1)^2(2x+1)(4x^2+1)(x^2+1)}$$

where

$g_2(x) = x^{n+4}(-4x^5+4x^3-x^2+1)W_{n+4}^2+x^{n+5}(-12x^4-6x^3+10x^2+6x+2)W_{n+3}^2+x^{n+5}(4x^4-6x^3-7x^2+6x+3)W_{n+2}^2+4x^{n+5}(-4x^5+4x^3-x^2+1)W_{n+1}^2-x^{n+3}(-16x^6-2x^5+14x^4+x^3+3x^2+x-1)W_{n+4}W_{n+3}+x^{n+4}(-6x^4-3x^3+5x^2+3x+1)W_{n+4}W_{n+2}+2x^{n+4}(8x^6+2x^5-6x^4-3x^3-3x^2+x+1)W_{n+4}W_{n+1}-x^{n+2}(8x^7-20x^6-11x^5+16x^4+x^3+5x^2+2x-1)W_{n+3}W_{n+2}+2x^{n+5}(-8x^5-6x^4+5x^3+3x^2+3x+3)W_{n+3}W_{n+1}-x^{n+1}(16x^9-4x^8-10x^7-21x^6-14x^5+21x^4+6x^3+5x^2+2x-1)W_{n+2}W_{n+1}-x^3(-4x^5+4x^3-x^2+1)W_3^2-x^4(-12x^4-6x^3+10x^2+6x+2)W_2^2-x^4(4x^4-6x^3-7x^2+6x+3)W_1^2-4x^4(-4x^5+4x^3-x^2+1)W_0^2+x^2(-16x^6-2x^5+14x^4+x^3+3x^2+x-1)W_3W_2-x^3(-6x^4-3x^3+5x^2+3x+1)W_3W_1-2x^3(8x^6+2x^5-6x^4-3x^3-3x^2+x+1)W_3W_0+x(8x^7-20x^6-11x^5+16x^4+x^3+5x^2+2x-1)W_2W_1-2x^4(-8x^5-6x^4+5x^3+3x^2+3x+3)W_2W_0+(16x^9-4x^8-10x^7-21x^6-14x^5+21x^4+6x^3+5x^2+2x-1)W_1W_0$

For $x=1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (b) using

$$\sum_{k=0}^{n} W_{k+1}W_k = \left.\frac{\frac{d^2}{dx^2}(g_2(x))}{\frac{d^2}{dx^2}((4x-1)(x-1)^2(x+1)^2(2x+1)(4x^2+1)(x^2+1))}\right|_{x=1}$$

(c) We use Theorem 3.1 (c). If we set $r=1,s=1,t=1,u=2$ in Theorem 3.1 (c) then we have

$$\sum_{k=0}^{n} x^k W_{k+2}W_k = \frac{g_3(x)}{(4x-1)(x-1)^2(x+1)^2(2x+1)(4x^2+1)(x^2+1)}$$

where

$g_3(x) = -x^{n+4}(2x^4-2)W_{n+4}^2+x^{n+3}(2x^5-x^4-2x+1)W_{n+3}^2-x^{n+5}(16x^5+4x^4-16x-4)W_{n+2}^2-4x^{n+5}(2x^4-2)W_{n+1}^2-x^{n+3}(x^4-1)W_{n+3}W_{n+4}+x^{n+2}(16x^8+4x^7+2x^6+2x^5-17x^4-4x^3-2x^2-2x+1)W_{n+4}W_{n+2}-2x^{n+4}(4x^5+x^4-4x-1)W_{n+4}W_{n+1}+x^{n+3}(-16x^7-4x^6+2x^5-x^4+16x^3+4x^2-2x+1)W_{n+3}W_{n+2}+x^{n+1}(24x^8+10x^7+3x^6+2x^5-25x^4-10x^3-3x^2-2x+1)W_{n+3}W_{n+1}-2x^{n+5}(8x^5+6x^4-8x-6)W_{n+2}W_{n+1}+x^3(2x^4-2)W_3^2-x^2(2x^5-x^4-2x+1)W_2^2+x^4(16x^5+4x^4-16x-4)W_1^2+4x^4(2x^4-2)W_0^2+x^2(x^4-1)W_3W_2-x(16x^8+4x^7+2x^6+2x^5-17x^4-4x^3-2x^2-2x+1)W_3W_1+2x^3(4x^5+x^4-4x-1)W_3W_0-x^2(-16x^7-4x^6+2x^5-x^4+16x^3+4x^2-2x+1)W_2W_1-(24x^8+10x^7+3x^6+2x^5-25x^4-10x^3-3x^2-2x+1)W_2W_0+2x^4(8x^5+6x^4-8x-6)W_1W_0$

For $x = 1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (c) using

$$\sum_{k=0}^{n} W_{k+2}W_k = \left.\frac{\frac{d^2}{dx^2}(g_3(x))}{\frac{d^2}{dx^2}((4x-1)(x-1)^2(x+1)^2(2x+1)(4x^2+1)(x^2+1))}\right|_{x=1}$$

(d) We use Theorem 3.1 (d). If we set $r = 1, s = 1, t = 1, u = 2$ in Theorem 3.1 (d) then we have

$$\sum_{k=0}^{n} x^k W_{k+3}W_k = \frac{g_4(x)}{(4x-1)(x-1)^2(x+1)^2(2x+1)(4x^2+1)(x^2+1)}$$

where

$g_4(x) = x^{n+4}(-x^3-4x^2+x+4)W_{n+4}^2 - x^{n+3}(16x^7+4x^6-14x^5-4x^4-6x^3+2x^2+4x-2)W_{n+3}^2 + x^{n+2}(-8x^7-2x^6+8x^5+6x^4+2x^3-5x^2-2x+1)W_{n+2}^2 + 4x^{n+5}(-x^3-4x^2+x+4)W_{n+1}^2 + x^{n+3}(16x^7+4x^6-14x^5-3x^4-3x^3-3x^2+x+2)W_{n+4}W_{n+3} + x^{n+2}(-8x^7-2x^6+7x^5+2x^4+3x^3-x^2-2x+1)W_{n+4}W_{n+2} - x^{n+1}(4x^7-15x^6-8x^5+11x^4+2x^3+5x^2+2x-1)W_{n+4}W_{n+1} + x^{n+2}(-16x^8+20x^7+20x^6-16x^5+6x^4-5x^3-11x^2+x+1)W_{n+3}W_{n+2} + 2x^{n+3}(-8x^7-2x^6+5x^5-6x^4+5x^3+7x^2-2x+1)W_{n+3}W_{n+1} - 2x^{n+2}(4x^7-13x^6+9x^4-6x^3+5x^2+2x-1)W_{n+2}W_{n+1} - x^3(-x^3-4x^2+x+4)W_3^2 + x^2(16x^7+4x^6-14x^5-4x^4-6x^3+2x^2+4x-2)W_2^2 - x(-8x^7-2x^6+8x^5+6x^4+2x^3-5x^2-2x+1)W_1^2 - 4x^4(-x^3-4x^2+x+4)W_0^2 - x^2(16x^7+4x^6-14x^5-3x^4-3x^3-3x^2+x+2)W_3W_2 - x(-8x^7-2x^6+7x^5+2x^4+3x^3-x^2-2x+1)W_3W_1 + (4x^7-15x^6-8x^5+11x^4+2x^3+5x^2+2x-1)W_3W_0 - x(-16x^8+20x^7+20x^6-16x^5+6x^4-5x^3-11x^2+x+1)W_2W_1 - 2x^2(-8x^7-2x^6+5x^5-6x^4+5x^3+7x^2-2x+1)W_2W_0 + 2x(4x^7-13x^6+9x^4-6x^3+5x^2+2x-1)W_1W_0$

For $x = 1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (d) using

$$\sum_{k=0}^{n} W_{k+3}W_k = \left.\frac{\frac{d^2}{dx^2}(g_4(x))}{\frac{d^2}{dx^2}((4x-1)(x-1)^2(x+1)^2(2x+1)(4x^2+1)(x^2+1))}\right|_{x=1}$$

From the last theorem, we have the following corollary which gives sum formulas of fourth order Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 1$).

Corollary 4.2. *For $n \geq 0$, fourth order Jacobsthal numbers have the following properties:*

(a) $\sum_{k=0}^{n} J_k^2 = \frac{1}{180}(14(n+8)J_{n+4}^2 + (38n+279)J_{n+3}^2 + 2(25n+164)J_{n+2}^2 + 4(14n+81)J_{n+1}^2 - (38n+293)J_{n+4}J_{n+3} - (26n+199)J_{n+4}J_{n+2} - 2(10n+79)J_{n+4}J_{n+1} - (2n-13)J_{n+3}J_{n+2} + 2(2n+27)J_{n+3}J_{n+1} + 4(4n+37)J_{n+2}J_{n+1} - 204)$.

(b) $\sum_{k=0}^{n} J_{k+1}J_k = \frac{1}{360}(-(10n+69)J_{n+4}^2 - 40(n+7)J_{n+3}^2 - (10n+51)J_{n+2}^2 - 4(10n+79)J_{n+1}^2 + 10(4n+29)J_{n+4}J_{n+3} - 20(n+6)J_{n+4}J_{n+2} + 8(5n+43)J_{n+4}J_{n+1} + 2(20n+109)J_{n+3}J_{n+2} - 4(20n+149)J_{n+3}J_{n+1} + 4(10n+23)J_{n+2}J_{n+1} + 12)$.

(c) $\sum_{k=0}^{n} J_{k+2}J_k = \frac{1}{180}(-2(2n+11)J_{n+4}^2 + (2n+13)J_{n+3}^2 - 4(10n+73)J_{n+2}^2 - 8(2n+13)J_{n+1}^2 - (2n+9)J_{n+4}J_{n+3} + (46n+325)J_{n+4}J_{n+2} - 2(10n+63)J_{n+4}J_{n+1} - (38n+279)J_{n+3}J_{n+2} + 2(38n+229)J_{n+3}J_{n+1} - 4(14n+99)J_{n+2}J_{n+1} + 228)$.

(d) $\sum_{k=0}^{n} J_{k+3}J_k = \frac{1}{360}(-(10n+47)J_{n+4}^2 - 8(5n+42)J_{n+3}^2 - (10n+101)J_{n+2}^2 - 4(10n+57)J_{n+1}^2 + 2(20n+173)J_{n+4}J_{n+3} - 4(5n+37)J_{n+4}J_{n+2} + 8(5n+23)J_{n+4}J_{n+1} + 2(20n+101)J_{n+3}J_{n+2} - 4(20n+141)J_{n+3}J_{n+1} + 20(2n+11)J_{n+2}J_{n+1} + 84)$.

Taking $W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10$ in the last theorem, we have the following corollary which presents sum formulas of fourth order Jacobsthal-Lucas numbers.

Corollary 4.3. *For $n \geq 0$, fourth order Jacobsthal-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} j_k^2 = \frac{1}{180}(14(n+8)j_{n+4}^2 + (38n+279)j_{n+3}^2 + 2(25n+164)j_{n+2}^2 + 4(14n+81)j_{n+1}^2 - (38n+293)j_{n+4}j_{n+3} - (26n+199)j_{n+4}j_{n+2} - 2(10n+79)j_{n+4}j_{n+1} - (2n-13)j_{n+3}j_{n+2} + 2(2n+27)j_{n+3}j_{n+1} + 4(4n+37)j_{n+2}j_{n+1} - 774)$.

(b) $\sum_{k=0}^{n} j_{k+1}j_k = \frac{1}{360}(-(10n+69)j_{n+4}^2 - 40(n+7)j_{n+3}^2 - (10n+51)j_{n+2}^2 - 4(10n+79)j_{n+1}^2 + 10(4n+29)j_{n+4}j_{n+3} - 20(n+6)j_{n+4}j_{n+2} + 8(5n+43)j_{n+4}j_{n+1} + 2(20n+109)j_{n+3}j_{n+2} - 4(20n+149)j_{n+3}j_{n+1} + 4(10n+23)j_{n+2}j_{n+1} - 369)$.

(c) $\sum_{k=0}^{n} j_{k+2}j_k = \frac{1}{180}(-2(2n+11)j_{n+4}^2 + (2n+13)j_{n+3}^2 - 4(10n+73)j_{n+2}^2 - 8(2n+13)j_{n+1}^2 - (2n+9)j_{n+4}j_{n+3} + (46n+325)j_{n+4}j_{n+2} - 2(10n+63)j_{n+4}j_{n+1} - (38n+279)j_{n+3}j_{n+2} + 2(38n+229)j_{n+3}j_{n+1} - 4(14n+99)j_{n+2}j_{n+1} - 126)$.

(d) $\sum_{k=0}^{n} j_{k+3}j_k = \frac{1}{360}(-(10n+47)j_{n+4}^2 - 8(5n+42)j_{n+3}^2 - (10n+101)j_{n+2}^2 - 4(10n+57)j_{n+1}^2 + 2(20n+173)j_{n+4}j_{n+3} - 4(5n+37)j_{n+4}j_{n+2} + 8(5n+23)j_{n+4}j_{n+1} + 2(20n+101)j_{n+3}j_{n+2} - 4(20n+141)j_{n+3}j_{n+1} + 20(2n+11)j_{n+2}j_{n+1} - 1287)$.

From the last theorem, we have the following corollary which gives sum formulas of modified fourth order Jacobsthal numbers (take $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3, K_3 = 10$).

Corollary 4.4. *For $n \geq 0$, modified fourth order Jacobsthal numbers have the following properties:*

(a) $\sum_{k=0}^{n} K_k^2 = \frac{1}{180}(14(n+8)K_{n+4}^2 + (38n+279)K_{n+3}^2 + 2(25n+164)K_{n+2}^2 + 4(14n+81)K_{n+1}^2 - (38n+293)K_{n+4}K_{n+3} - (26n+199)K_{n+4}K_{n+2} - 2(10n+79)K_{n+4}K_{n+1} - (2n-13)K_{n+3}K_{n+2} + 2(2n+27)K_{n+3}K_{n+1} + 4(4n+37)K_{n+2}K_{n+1} - 2030)$.

(b) $\sum_{k=0}^{n} K_{k+1}K_k = \frac{1}{360}(-(10n+69)K_{n+4}^2 - 40(n+7)K_{n+3}^2 - (10n+51)K_{n+2}^2 - 4(10n+79)K_{n+1}^2 + 10(4n+29)K_{n+4}K_{n+3} - 20(n+6)K_{n+4}K_{n+2} + 8(5n+43)K_{n+4}K_{n+1} + 2(20n+109)K_{n+3}K_{n+2} - 4(20n+149)K_{n+3}K_{n+1} + 4(10n+23)K_{n+2}K_{n+1} - 1081)$.

(c) $\sum_{k=0}^{n} K_{k+2}K_k = \frac{1}{180}(-2(2n+11)K_{n+4}^2 + (2n+13)K_{n+3}^2 - 4(10n+73)K_{n+2}^2 - 8(2n+13)K_{n+1}^2 - (2n+9)K_{n+4}K_{n+3} + (46n+325)K_{n+4}K_{n+2} - 2(10n+63)K_{n+4}K_{n+1} - (38n+279)K_{n+3}K_{n+2} + 2(38n+229)K_{n+3}K_{n+1} - 4(14n+99)K_{n+2}K_{n+1} + 1650)$.

(d) $\sum_{k=0}^{n} K_{k+3}K_k = \frac{1}{360}(-(10n+47)K_{n+4}^2 - 8(5n+42)K_{n+3}^2 - (10n+101)K_{n+2}^2 - 4(10n+57)K_{n+1}^2 + 2(20n+173)K_{n+4}K_{n+3} - 4(5n+37)K_{n+4}K_{n+2} + 8(5n+23)K_{n+4}K_{n+1} + 2(20n+101)K_{n+3}K_{n+2} - 4(20n+141)K_{n+3}K_{n+1} + 20(2n+11)K_{n+2}K_{n+1} - 743)$.

4.2 The case $x = -1$

In this subsection we consider the special case $x = -1$.

In this section, we present the closed form solutions (identities) of the sums $\sum_{k=0}^{n}(-1)^k W_k^2$, $\sum_{k=0}^{n}(-1)^k W_{k+1}W_k$, $\sum_{k=0}^{n}(-1)^k W_{k+2}W_k$ and $\sum_{k=0}^{n}(-1)^k W_{k+3}W_k$ for the specific case of the sequence $\{W_n\}$.

Taking $r = s = t = u = 1$ in Theorem 3.1, we obtain the following proposition.

Proposition 4.1. *If $r = s = t = u = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n}(-1)^k W_k^2 = (-1)^n(-W_{n+2}^2 + W_{n+1}^2 + W_{n+4}W_{n+2} - W_{n+3}W_{n+2} - W_{n+3}W_{n+1}) - W_1^2 + W_0^2 + W_3W_1 - W_2W_1 - W_2W_0$.

(b) $\sum_{k=0}^{n}(-1)^k W_{k+1}W_k = \frac{1}{2}((-1)^n(-W_{n+3}^2 + W_{n+2}^2 + W_{n+4}W_{n+3} - W_{n+4}W_{n+1} - 2W_{n+3}W_{n+2} + 3W_{n+2}W_{n+1}) - W_2^2 + W_1^2 + W_3W_2 - W_3W_0 - 2W_2W_1 + 3W_1W_0)$.

(c) $\sum_{k=0}^{n}(-1)^k W_{k+2}W_k = \frac{1}{2}((-1)^n(W_{n+4}^2 - W_{n+3}^2 + W_{n+2}^2 - W_{n+1}^2 - W_{n+4}W_{n+3} - W_{n+4}W_{n+2} - W_{n+4}W_{n+1} + W_{n+3}W_{n+2} + W_{n+3}W_{n+1} - W_{n+2}W_{n+1}) + W_3^2 - W_2^2 + W_1^2 - W_0^2 - W_3W_2 - W_3W_1 - W_3W_0 + W_2W_1 + W_2W_0 - W_1W_0)$.

(d) $\sum_{k=0}^{n}(-1)^k W_{k+3}W_k = \frac{1}{2}((-1)^n (W_{n+4}^2 - W_{n+3}^2 - W_{n+2}^2 - W_{n+1}^2 - W_{n+4}W_{n+3} + W_{n+4}W_{n+2} + W_{n+4}W_{n+1} - W_{n+3}W_{n+2} - 3W_{n+3}W_{n+1} - 3W_{n+2}W_{n+1}) + W_3^2 - W_2^2 - W_1^2 - W_0^2 - W_3W_2 + W_3W_1 + W_3W_0 - W_2W_1 - 3W_2W_0 - 3W_1W_0)$.

From the above proposition, we have the following corollary which gives sum formulas of Tetranacci numbers (take $W_n = M_n$ with $M_0 = 0, M_1 = 1, M_2 = 1, M_3 = 2$).

Corollary 4.5. *For $n \geq 0$, Tetranacci numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k M_k^2 = (-1)^n (-M_{n+2}^2 + M_{n+1}^2 + M_{n+4}M_{n+2} - M_{n+3}M_{n+2} - M_{n+3}M_{n+1})$.

(b) $\sum_{k=0}^{n}(-1)^k M_{k+1}M_k = \frac{1}{2}(-1)^n (-M_{n+3}^2 + M_{n+2}^2 + M_{n+4}M_{n+3} - M_{n+4}M_{n+1} - 2M_{n+3}M_{n+2} + 3M_{n+2}M_{n+1})$.

(c) $\sum_{k=0}^{n}(-1)^k M_{k+2}M_k = \frac{1}{2}((-1)^n (M_{n+4}^2 - M_{n+3}^2 + M_{n+2}^2 - M_{n+1}^2 - M_{n+4}M_{n+3} - M_{n+4}M_{n+2} - M_{n+4}M_{n+1} + M_{n+3}M_{n+2} + M_{n+3}M_{n+1} - M_{n+2}M_{n+1}) + 1)$.

(d) $\sum_{k=0}^{n}(-1)^k M_{k+3}M_k = \frac{1}{2}((-1)^n (M_{n+4}^2 - M_{n+3}^2 - M_{n+2}^2 - M_{n+1}^2 - M_{n+4}M_{n+3} + M_{n+4}M_{n+2} + M_{n+4}M_{n+1} - M_{n+3}M_{n+2} - 3M_{n+3}M_{n+1} - 3M_{n+2}M_{n+1}) + 1)$.

Taking $W_n = R_n$ with $R_0 = 4, R_1 = 1, R_2 = 3, R_3 = 7$ in the above proposition, we have the following corollary which presents sum formulas of Tetranacci-Lucas numbers.

Corollary 4.6. *For $n \geq 0$, Tetranacci-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k R_k^2 = (-1)^n (-R_{n+2}^2 + R_{n+1}^2 + R_{n+4}R_{n+2} - R_{n+3}R_{n+2} - R_{n+3}R_{n+1}) + 7$.

(b) $\sum_{k=0}^{n}(-1)^k R_{k+1}R_k = \frac{1}{2}((-1)^n (-R_{n+3}^2 + R_{n+2}^2 + R_{n+4}R_{n+3} - R_{n+4}R_{n+1} - 2R_{n+3}R_{n+2} + 3R_{n+2}R_{n+1}) - 9)$.

(c) $\sum_{k=0}^{n}(-1)^k R_{k+2}R_k = \frac{1}{2}((-1)^n (R_{n+4}^2 - R_{n+3}^2 + R_{n+2}^2 - R_{n+1}^2 - R_{n+4}R_{n+3} - R_{n+4}R_{n+2} - R_{n+4}R_{n+1} + R_{n+3}R_{n+2} + R_{n+3}R_{n+1} - R_{n+2}R_{n+1}) - 20)$.

(d) $\sum_{k=0}^{n}(-1)^k R_{k+3}R_k = \frac{1}{2}((-1)^n (R_{n+4}^2 - R_{n+3}^2 - R_{n+2}^2 - R_{n+1}^2 - R_{n+4}R_{n+3} + R_{n+4}R_{n+2} + R_{n+4}R_{n+1} - R_{n+3}R_{n+2} - 3R_{n+3}R_{n+1} - 3R_{n+2}R_{n+1}) - 14)$.

Taking $r = 2, s = 1, t = 1, u = 1$ in Theorem 3.1, we obtain the following proposition.

Proposition 4.2. *If $r = 2, s = 1, t = 1, u = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n}(-1)^k W_k^2 = \frac{1}{14}((-1)^n (W_{n+4}^2 + 7W_{n+3}^2 - 16W_{n+2}^2 + 13W_{n+1}^2 - 6W_{n+4}W_{n+3} + 8W_{n+4}W_{n+2} + 2W_{n+4}W_{n+1} - 8W_{n+3}W_{n+2} - 10W_{n+3}W_{n+1} - 4W_{n+2}W_{n+1}) + W_3^2 + 7W_2^2 - 16W_1^2 + 13W_0^2 - 6W_3W_2 + 8W_3W_1 + 2W_3W_0 - 8W_2W_1 - 10W_2W_0 - 4W_1W_0)$.

(b) $\sum_{k=0}^{n}(-1)^k W_{k+1}W_k = \frac{1}{14}((-1)^n (-W_{n+4}^2 - 7W_{n+3}^2 + 2W_{n+2}^2 + W_{n+1}^2 + 6W_{n+4}W_{n+3} + 6W_{n+4}W_{n+2} - 2W_{n+4}W_{n+1} - 20W_{n+3}W_{n+2} - 4W_{n+3}W_{n+1} + 18W_{n+2}W_{n+1}) - W_3^2 - 7W_2^2 + 2W_1^2 + W_0^2 + 6W_3W_2 + 6W_3W_1 - 2W_3W_0 - 20W_2W_1 - 4W_2W_0 + 18W_1W_0)$.

(c) $\sum_{k=0}^{n}(-1)^k W_{k+2}W_k = \frac{1}{14}((-1)^n (3W_{n+4}^2 - 7W_{n+3}^2 + 8W_{n+2}^2 - 3W_{n+1}^2 - 4W_{n+4}W_{n+3} - 4W_{n+4}W_{n+2} - 8W_{n+4}W_{n+1} + 4W_{n+3}W_{n+2} + 12W_{n+3}W_{n+1} + 2W_{n+2}W_{n+1}) + 3W_3^2 - 7W_2^2 + 8W_1^2 - 3W_0^2 - 4W_3W_2 - 4W_3W_1 - 8W_3W_0 + 4W_2W_1 + 12W_2W_0 + 2W_1W_0)$.

(d) $\sum_{k=0}^{n}(-1)^k W_{k+3}W_k = \frac{1}{2}((-1)^n (W_{n+4}^2 - W_{n+3}^2 - W_{n+1}^2 - 2W_{n+4}W_{n+3} - 2W_{n+3}W_{n+1} - 2W_{n+2}W_{n+1}) + W_3^2 - W_2^2 - W_0^2 - 2W_3W_2 - 2W_2W_0 - 2W_1W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of fourth-order Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 5$).

Corollary 4.7. *For $n \geq 0$, fourth-order Pell numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k P_k^2 = \frac{1}{14}((-1)^n (P_{n+4}^2 + 7P_{n+3}^2 - 16P_{n+2}^2 + 13P_{n+1}^2 - 6P_{n+4}P_{n+3} + 8P_{n+4}P_{n+2} + 2P_{n+4}P_{n+1} - 8P_{n+3}P_{n+2} - 10P_{n+3}P_{n+1} - 4P_{n+2}P_{n+1}) + 1)$.

(b) $\sum_{k=0}^{n}(-1)^k P_{k+1}P_k = \frac{1}{14}((-1)^n(-P_{n+4}^2 - 7P_{n+3}^2 + 2P_{n+2}^2 + P_{n+1}^2 + 6P_{n+4}P_{n+3} + 6P_{n+4}P_{n+2} - 2P_{n+4}P_{n+1} - 20P_{n+3}P_{n+2} - 4P_{n+3}P_{n+1} + 18P_{n+2}P_{n+1}) - 1)$.

(c) $\sum_{k=0}^{n}(-1)^k P_{k+2}P_k = \frac{1}{14}((-1)^n(3P_{n+4}^2 - 7P_{n+3}^2 + 8P_{n+2}^2 - 3P_{n+1}^2 - 4P_{n+4}P_{n+3} - 4P_{n+4}P_{n+2} - 8P_{n+4}P_{n+1} + 4P_{n+3}P_{n+2} + 12P_{n+3}P_{n+1} + 2P_{n+2}P_{n+1}) + 3)$.

(d) $\sum_{k=0}^{n}(-1)^k P_{k+3}P_k = \frac{1}{2}((-1)^n(P_{n+4}^2 - P_{n+3}^2 - P_{n+1}^2 - 2P_{n+4}P_{n+3} - 2P_{n+3}P_{n+1} - 2P_{n+2}P_{n+1}) + 1)$.

Taking $W_n = Q_n$ with $Q_0 = 4, Q_1 = 2, Q_2 = 6, Q_3 = 17$ in the last proposition, we have the following corollary which presents sum formulas of fourth-order Pell-Lucas numbers.

Corollary 4.8. *For $n \geq 0$, fourth-order Pell-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k Q_k^2 = \frac{1}{14}((-1)^n(Q_{n+4}^2 + 7Q_{n+3}^2 - 16Q_{n+2}^2 + 13Q_{n+1}^2 - 6Q_{n+4}Q_{n+3} + 8Q_{n+4}Q_{n+2} + 2Q_{n+4}Q_{n+1} - 8Q_{n+3}Q_{n+2} - 10Q_{n+3}Q_{n+1} - 4Q_{n+2}Q_{n+1}) + 113)$.

(b) $\sum_{k=0}^{n}(-1)^k Q_{k+1}Q_k = \frac{1}{14}((-1)^n(-Q_{n+4}^2 - 7Q_{n+3}^2 + 2Q_{n+2}^2 + Q_{n+1}^2 + 6Q_{n+4}Q_{n+3} + 6Q_{n+4}Q_{n+2} - 2Q_{n+4}Q_{n+1} - 20Q_{n+3}Q_{n+2} - 4Q_{n+3}Q_{n+1} + 18Q_{n+2}Q_{n+1}) - 29)$.

(c) $\sum_{k=0}^{n}(-1)^k Q_{k+2}Q_k = \frac{1}{14}((-1)^n(3Q_{n+4}^2 - 7Q_{n+3}^2 + 8Q_{n+2}^2 - 3Q_{n+1}^2 - 4Q_{n+4}Q_{n+3} - 4Q_{n+4}Q_{n+2} - 8Q_{n+4}Q_{n+1} + 4Q_{n+3}Q_{n+2} + 12Q_{n+3}Q_{n+1} + 2Q_{n+2}Q_{n+1}) - 137)$.

(d) $\sum_{k=0}^{n}(-1)^k Q_{k+3}Q_k = \frac{1}{2}((-1)^n(Q_{n+4}^2 - Q_{n+3}^2 - Q_{n+1}^2 - 2Q_{n+4}Q_{n+3} - 2Q_{n+3}Q_{n+1} - 2Q_{n+2}Q_{n+1}) - 31)$.

From the last proposition, we have the following corollary which gives sum formulas of modified fourth-order Pell numbers (take $W_n = E_n$ with $E_0 = 0, E_1 = 1, E_2 = 1, E_3 = 3$).

Corollary 4.9. *For $n \geq 0$, modified fourth-order Pell numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k E_k^2 = \frac{1}{14}((-1)^n(E_{n+4}^2 + 7E_{n+3}^2 - 16E_{n+2}^2 + 13E_{n+1}^2 - 6E_{n+4}E_{n+3} + 8E_{n+4}E_{n+2} + 2E_{n+4}E_{n+1} - 8E_{n+3}E_{n+2} - 10E_{n+3}E_{n+1} - 4E_{n+2}E_{n+1}) - 2)$.

(b) $\sum_{k=0}^{n}(-1)^k E_{k+1}E_k = \frac{1}{14}((-1)^n(-E_{n+4}^2 - 7E_{n+3}^2 + 2E_{n+2}^2 + E_{n+1}^2 + 6E_{n+4}E_{n+3} + 6E_{n+4}E_{n+2} - 2E_{n+4}E_{n+1} - 20E_{n+3}E_{n+2} - 4E_{n+3}E_{n+1} + 18E_{n+2}E_{n+1}) + 2)$.

(c) $\sum_{k=0}^{n}(-1)^k E_{k+2}E_k = \frac{1}{14}((-1)^n(3E_{n+4}^2 - 7E_{n+3}^2 + 8E_{n+2}^2 - 3E_{n+1}^2 - 4E_{n+4}E_{n+3} - 4E_{n+4}E_{n+2} - 8E_{n+4}E_{n+1} + 4E_{n+3}E_{n+2} + 12E_{n+3}E_{n+1} + 2E_{n+2}E_{n+1}) + 8)$.

(d) $\sum_{k=0}^{n}(-1)^k E_{k+3}E_k = \frac{1}{2}((-1)^n(E_{n+4}^2 - E_{n+3}^2 - E_{n+1}^2 - 2E_{n+4}E_{n+3} - 2E_{n+3}E_{n+1} - 2E_{n+2}E_{n+1}) + 2)$.

Observe that setting $x = -1, r = 1, s = 1, t = 1, u = 2$ (i.e. for the generalized fourth order Jacobsthal case) in Theorem 3.1 (a),(b),(c) and (d) makes the right hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule (using twice) however provides the evaluation of the sum formulas.

Taking $r = 1, s = 1, t = 1, u = 2$ in Theorem 3.1, we obtain the following theorem.

Theorem 4.10. *If $r = 1, s = 1, t = 1, u = 2$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n}(-1)^k W_k^2 = \frac{1}{100}((-1)^n((4n+41)W_{n+4}^2 - (6n+47)W_{n+3}^2 + 3(8n+59)W_{n+2}^2 - 16(n+5)W_{n+1}^2 - (2n+31)W_{n+4}W_{n+3} - (22n+177)W_{n+4}W_{n+2} - 6(2n+19)W_{n+4}W_{n+1} + (18n+155)W_{n+3}W_{n+2} + 2(14n+109)W_{n+3}W_{n+1} + 8(n+9)W_{n+2}W_{n+1}) + 37W_3^2 - 41W_2^2 + 153W_1^2 - 64W_0^2 - 29W_3W_2 - 155W_3W_1 - 102W_3W_0 + 137W_2W_1 + 190W_2W_0 + 64W_1W_0)$.

(b) $\sum_{k=0}^{n}(-1)^k W_{k+1}W_k = \frac{1}{200}((-1)^n(3(2n+17)W_{n+4}^2 + 4(4n+31)W_{n+3}^2 - 7(2n+15)W_{n+2}^2 - 12(2n+19)W_{n+1}^2 - 4(7n+55)W_{n+4}W_{n+3} - 2(4n+27)W_{n+4}W_{n+2} + 4(8n+67)W_{n+4}W_{n+1} + 52(n+7)W_{n+3}W_{n+2} - 8(n+13)W_{n+3}W_{n+1} - 4(22n+157)W_{n+2}W_{n+1}) + 45W_3^2 + 108W_2^2 - 91W_1^2 - 204W_0^2 - 192W_3W_2 - 46W_3W_1 + 236W_3W_0 + 312W_2W_1 - 96W_2W_0 - 540W_1W_0)$.

(c) $\sum_{k=0}^{n}(-1)^k W_{k+2}W_k = \frac{1}{100}((-1)^n(-2(2n+11)W_{n+4}^2 + (6n+31)W_{n+3}^2 - 4(6n+47)W_{n+2}^2 + 8(2n+13)W_{n+1}^2 + (2n+9)W_{n+4}W_{n+3} + (22n+185)W_{n+4}W_{n+2} + 2(6n+41)W_{n+4}W_{n+1} - (18n+157)W_{n+3}W_{n+2} - 14(2n+15)W_{n+3}W_{n+1} - 4(2n+21)W_{n+2}W_{n+1}) - 18W_3^2 + 25W_2^2 - 164W_1^2 + 88W_0^2 - 76W_1W_0 - 182W_2W_0 + 70W_3W_0 - 139W_2W_1 + 163W_3W_1 + 7W_3W_2)$.

(d) $\sum_{k=0}^{n}(-1)^k W_{k+3}W_k = \frac{1}{200}((-1)^n(-(6n+25)W_{n+4}^2 - 4(4n+47)W_{n+3}^2 + (14n+111)W_{n+2}^2 + 4(6n+31)W_{n+1}^2 + 4(7n+58)W_{n+4}W_{n+3} + 2(4n+43)W_{n+4}W_{n+2} - 4(8n+49)W_{n+4}W_{n+1} - 4(13n+118)W_{n+3}W_{n+2} + 8(n-8)W_{n+3}W_{n+1} + 4(22n+145)W_{n+2}W_{n+1}) - 19W_3^2 - 172W_2^2 + 97W_1^2 + 100W_0^2 + 204W_3W_2 + 78W_3W_1 - 164W_3W_0 - 420W_2W_1 - 72W_2W_0 + 492W_1W_0)$.

Proof.

(a) From the proof of Theorem 4.1 (a) we know that if $r = 1, s = 1, t = 1, u = 2$, then

$$\sum_{k=0}^{n} x^k W_k^2 = \frac{g_1(x)}{(4x-1)(x-1)^2(x+1)^2(2x+1)(4x^2+1)(x^2+1)}.$$

For $x = -1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (a) using

$$\sum_{k=0}^{n} W_k^2 = \left.\frac{\frac{d^2}{dx^2}(g_1(x))}{\frac{d^2}{dx^2}((4x-1)(x-1)^2(x+1)^2(2x+1)(4x^2+1)(x^2+1))}\right|_{x=-1}.$$

(b) From the proof of Theorem 4.1 (b) we know that if $r = 1, s = 1, t = 1, u = 2$, then

$$\sum_{k=0}^{n} x^k W_{k+1}W_k = \frac{g_2(x)}{(4x-1)(x-1)^2(x+1)^2(2x+1)(4x^2+1)(x^2+1)}.$$

For $x = -1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (b) using

$$\sum_{k=0}^{n} W_{k+1}W_k = \left.\frac{\frac{d^2}{dx^2}(g_2(x))}{\frac{d^2}{dx^2}((4x-1)(x-1)^2(x+1)^2(2x+1)(4x^2+1)(x^2+1))}\right|_{x=-1}.$$

(c) From the proof of Theorem 4.1 (c) we know that if $r = 1, s = 1, t = 1, u = 2$, then

$$\sum_{k=0}^{n} x^k W_{k+2}W_k = \frac{g_3(x)}{(4x-1)(x-1)^2(x+1)^2(2x+1)(4x^2+1)(x^2+1)}.$$

For $x = -1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (c) using

$$\sum_{k=0}^{n} W_{k+2}W_k = \left.\frac{\frac{d^2}{dx^2}(g_3(x))}{\frac{d^2}{dx^2}((4x-1)(x-1)^2(x+1)^2(2x+1)(4x^2+1)(x^2+1))}\right|_{x=-1}.$$

(d) From the proof of Theorem 4.1 (d) we know that if $r = 1, s = 1, t = 1, u = 2$, then

$$\sum_{k=0}^{n} x^k W_{k+3}W_k = \frac{g_4(x)}{(4x-1)(x-1)^2(x+1)^2(2x+1)(4x^2+1)(x^2+1)}.$$

For $x = -1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (d) using

$$\sum_{k=0}^{n} W_{k+3}W_k = \left.\frac{\frac{d^2}{dx^2}(g_4(x))}{\frac{d^2}{dx^2}((4x-1)(x-1)^2(x+1)^2(2x+1)(4x^2+1)(x^2+1))}\right|_{x=-1}.$$

From the last theorem, we have the following corollary which gives sum formulas of fourth order Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 1$).

Corollary 4.11. *For $n \geq 0$, fourth order Jacobsthal numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k J_k^2 = \frac{1}{100}((-1)^n((4n+41)J_{n+4}^2-(6n+47)J_{n+3}^2+3(8n+59)J_{n+2}^2-16(n+5)J_{n+1}^2-(2n+31)J_{n+4}J_{n+3}-(22n+177)J_{n+4}J_{n+2}-6(2n+19)J_{n+4}J_{n+1}+(18n+155)J_{n+3}J_{n+2}+2(14n+109)J_{n+3}J_{n+1}+8(n+9)J_{n+2}J_{n+1})+102)$.

(b) $\sum_{k=0}^{n}(-1)^k J_{k+1}J_k = \frac{1}{200}((-1)^n(3(2n+17)J_{n+4}^2+4(4n+31)J_{n+3}^2-7(2n+15)J_{n+2}^2-12(2n+19)J_{n+1}^2-4(7n+55)J_{n+4}J_{n+3}-2(4n+27)J_{n+4}J_{n+2}+4(8n+67)J_{n+4}J_{n+1}+52(n+7)J_{n+3}J_{n+2}-8(n+13)J_{n+3}J_{n+1}-4(22n+157)J_{n+2}J_{n+1})+136)$.

(c) $\sum_{k=0}^{n}(-1)^k J_{k+2}J_k = \frac{1}{100}((-1)^n(-2(2n+11)J_{n+4}^2+(6n+31)J_{n+3}^2-4(6n+47)J_{n+2}^2+8(2n+13)J_{n+1}^2+(2n+9)J_{n+4}J_{n+3}+(22n+185)J_{n+4}J_{n+2}+2(6n+41)J_{n+4}J_{n+1}-(18n+157)J_{n+3}J_{n+2}-14(2n+15)J_{n+3}J_{n+1}-4(2n+21)J_{n+2}J_{n+1})-126)$.

(d) $\sum_{k=0}^{n}(-1)^k J_{k+3}J_k = \frac{1}{200}((-1)^n(-(6n+25)J_{n+4}^2-4(4n+47)J_{n+3}^2+(14n+111)J_{n+2}^2+4(6n+31)J_{n+1}^2+4(7n+58)J_{n+4}J_{n+3}+2(4n+43)J_{n+4}J_{n+2}-4(8n+49)J_{n+4}J_{n+1}-4(13n+118)J_{n+3}J_{n+2}+8(n-8)J_{n+3}J_{n+1}+4(22n+145)J_{n+2}J_{n+1})-232)$.

Taking $W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10$ in the last theorem, we have the following corollary which presents sum formulas of fourth order Jacobsthal-Lucas numbers.

Corollary 4.12. *For $n \geq 0$, fourth order Jacobsthal-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k j_k^2 = \frac{1}{100}((-1)^n((4n+41)j_{n+4}^2-(6n+47)j_{n+3}^2+3(8n+59)j_{n+2}^2-16(n+5)j_{n+1}^2-(2n+31)j_{n+4}j_{n+3}-(22n+177)j_{n+4}j_{n+2}-6(2n+19)j_{n+4}j_{n+1}+(18n+155)j_{n+3}j_{n+2}+2(14n+109)j_{n+3}j_{n+1}+8(n+9)j_{n+2}j_{n+1})+245)$.

(b) $\sum_{k=0}^{n}(-1)^k j_{k+1}j_k = \frac{1}{200}((-1)^n(3(2n+17)j_{n+4}^2+4(4n+31)j_{n+3}^2-7(2n+15)j_{n+2}^2-12(2n+19)j_{n+1}^2-4(7n+55)j_{n+4}j_{n+3}-2(4n+27)j_{n+4}j_{n+2}+4(8n+67)j_{n+4}j_{n+1}+52(n+7)j_{n+3}j_{n+2}-8(n+13)j_{n+3}j_{n+1}-4(22n+157)j_{n+2}j_{n+1})+473)$.

(c) $\sum_{k=0}^{n}(-1)^k j_{k+2}j_k = \frac{1}{100}((-1)^n(-2(2n+11)j_{n+4}^2+(6n+31)j_{n+3}^2-4(6n+47)j_{n+2}^2+8(2n+13)j_{n+1}^2+(2n+9)j_{n+4}j_{n+3}+(22n+185)j_{n+4}j_{n+2}+2(6n+41)j_{n+4}j_{n+1}-(18n+157)j_{n+3}j_{n+2}-14(2n+15)j_{n+3}j_{n+1}-4(2n+21)j_{n+2}j_{n+1})-274)$.

(d) $\sum_{k=0}^{n}(-1)^k j_{k+3}j_k = \frac{1}{200}((-1)^n(-(6n+25)j_{n+4}^2-4(4n+47)j_{n+3}^2+(14n+111)j_{n+2}^2+4(6n+31)j_{n+1}^2+4(7n+58)j_{n+4}j_{n+3}+2(4n+43)j_{n+4}j_{n+2}-4(8n+49)j_{n+4}j_{n+1}-4(13n+118)j_{n+3}j_{n+2}+8(n-8)j_{n+3}j_{n+1}+4(22n+145)j_{n+2}j_{n+1})+161)$.

From the last theorem, we have the following corollary which gives sum formulas of modified fourth order Jacobsthal numbers (take $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3, K_3 = 10$).

Corollary 4.13. *For $n \geq 0$, modified fourth order Jacobsthal numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k K_k^2 = \frac{1}{100}((-1)^n((4n+41)K_{n+4}^2-(6n+47)K_{n+3}^2+3(8n+59)K_{n+2}^2-16(n+5)K_{n+1}^2-(2n+31)K_{n+4}K_{n+3}-(22n+177)K_{n+4}K_{n+2}-6(2n+19)K_{n+4}K_{n+1}+(18n+155)K_{n+3}K_{n+2}+2(14n+109)K_{n+3}K_{n+1}+8(n+9)K_{n+2}K_{n+1})-259)$.

(b) $\sum_{k=0}^{n}(-1)^k K_{k+1}K_k = \frac{1}{200}((-1)^n(3(2n+17)K_{n+4}^2+4(4n+31)K_{n+3}^2-7(2n+15)K_{n+2}^2-12(2n+19)K_{n+1}^2-4(7n+55)K_{n+4}K_{n+3}-2(4n+27)K_{n+4}K_{n+2}+4(8n+67)K_{n+4}K_{n+1}+52(n+7)K_{n+3}K_{n+2}-8(n+13)K_{n+3}K_{n+1}-4(22n+157)K_{n+2}K_{n+1})+2857)$.

(c) $\sum_{k=0}^{n}(-1)^k K_{k+2}K_k = \frac{1}{100}((-1)^n(-2(2n+11)K_{n+4}^2+(6n+31)K_{n+3}^2-4(6n+47)K_{n+2}^2+8(2n+13)K_{n+1}^2+(2n+9)K_{n+4}K_{n+3}+(22n+185)K_{n+4}K_{n+2}+2(6n+41)K_{n+4}K_{n+1}-(18n+157)K_{n+3}K_{n+2}-14(2n+15)K_{n+3}K_{n+1}-4(2n+21)K_{n+2}K_{n+1})+710)$.

(d) $\sum_{k=0}^{n}(-1)^k K_{k+3}K_k = \frac{1}{200}((-1)^n(-(6n+25)K_{n+4}^2 - 4(4n+47)K_{n+3}^2 + (14n+111)K_{n+2}^2 + 4(6n+31)K_{n+1}^2 + 4(7n+58)K_{n+4}K_{n+3} + 2(4n+43)K_{n+4}K_{n+2} - 4(8n+49)K_{n+4}K_{n+1} - 4(13n+118)K_{n+3}K_{n+2} + 8(n-8)K_{n+3}K_{n+1} + 4(22n+145)K_{n+2}K_{n+1}) - 903)$.

Taking $r = 2, s = 3, t = 5$ in Theorem 3.1, we obtain the following proposition.

Proposition 4.3. *If $r = 2, s = 3, t = 5$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n}(-1)^k W_k^2 = \frac{1}{1278}((-1)^n(13W_{n+4}^2 - 37W_{n+3}^2 + 364W_{n+2}^2 + 641W_{n+1}^2 - 14W_{n+4}W_{n+3} - 148W_{n+4}W_{n+2} - 266W_{n+4}W_{n+1} + 196W_{n+3}W_{n+2} + 490W_{n+3}W_{n+1} + 952W_{n+2}W_{n+1}) + 13W_3^2 - 37W_2^2 + 364W_1^2 + 641W_0^2 - 14W_3W_2 - 148W_3W_1 - 266W_3W_0 + 196W_2W_1 + 490W_2W_0 + 952W_1W_0)$.

(b) $\sum_{k=0}^{n}(-1)^k W_{k+1}W_k = \frac{1}{1278}((-1)^n(19W_{n+4}^2 + 77W_{n+3}^2 + 106W_{n+2}^2 - 931W_{n+1}^2 - 86W_{n+4}W_{n+3} - 118W_{n+4}W_{n+2} + 70W_{n+4}W_{n+1} + 352W_{n+3}W_{n+2} + 28W_{n+3}W_{n+1} - 542W_{n+2}W_{n+1}) + 19W_3^2 + 77W_2^2 + 106W_1^2 - 931W_0^2 - 86W_3W_2 - 118W_3W_1 + 70W_3W_0 + 352W_2W_1 + 28W_2W_0 - 542W_1W_0)$.

(c) $\sum_{k=0}^{n}(-1)^k W_{k+2}W_k = \frac{1}{426}((-1)^n(W_{n+4}^2 + 19W_{n+3}^2 - 256W_{n+2}^2 - 49W_{n+1}^2 - 12W_{n+4}W_{n+3} + 76W_{n+4}W_{n+2} + 56W_{n+4}W_{n+1} - 116W_{n+3}W_{n+2} - 148W_{n+3}W_{n+1} - 462W_{n+2}W_{n+1}) + W_3^2 + 19W_2^2 - 256W_1^2 - 49W_0^2 - 12W_3W_2 + 76W_3W_1 + 56W_3W_0 - 148W_2W_0 - 116W_2W_1 - 462W_1W_0)$.

(d) $\sum_{k=0}^{n}(-1)^k W_{k+3}W_k = \frac{1}{1278}((-1)^n(-5W_{n+4}^2 - 379W_{n+3}^2 - 140W_{n+2}^2 + 245W_{n+1}^2 + 202W_{n+4}W_{n+3} + 188W_{n+4}W_{n+2} + 4W_{n+4}W_{n+1} - 1124W_{n+3}W_{n+2} - 1106W_{n+3}W_{n+1} + 322W_{n+2}W_{n+1}) - 5W_3^2 - 379W_2^2 - 140W_1^2 + 245W_0^2 + 202W_3W_2 + 188W_3W_1 + 4W_3W_0 - 1124W_2W_1 - 1106W_2W_0 + 322W_1W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of 4-primes numbers (take $W_n = G_n$ with $G_0 = 0, G_1 = 0, G_2 = 1, G_3 = 2$).

Corollary 4.14. *For $n \geq 0$, 4-primes numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k G_k^2 = \frac{1}{1278}((-1)^n(13G_{n+4}^2 - 37G_{n+3}^2 + 364G_{n+2}^2 + 641G_{n+1}^2 - 14G_{n+4}G_{n+3} - 148G_{n+4}G_{n+2} - 266G_{n+4}G_{n+1} + 196G_{n+3}G_{n+2} + 490G_{n+3}G_{n+1} + 952G_{n+2}G_{n+1}) - 13)$.

(b) $\sum_{k=0}^{n}(-1)^k G_{k+1}G_k = \frac{1}{1278}((-1)^n(19G_{n+4}^2 + 77G_{n+3}^2 + 106G_{n+2}^2 - 931G_{n+1}^2 - 86G_{n+4}G_{n+3} - 118G_{n+4}G_{n+2} + 70G_{n+4}G_{n+1} + 352G_{n+3}G_{n+2} + 28G_{n+3}G_{n+1} - 542G_{n+2}G_{n+1}) - 19)$.

(d) $\sum_{k=0}^{n}(-1)^k G_{k+2}G_k = \frac{1}{426}((-1)^n(G_{n+4}^2 + 19G_{n+3}^2 - 256G_{n+2}^2 - 49G_{n+1}^2 - 12G_{n+4}G_{n+3} + 76G_{n+4}G_{n+2} + 56G_{n+4}G_{n+1} - 116G_{n+3}G_{n+2} - 148G_{n+3}G_{n+1} - 462G_{n+2}G_{n+1}) - 1)$.

(c) $\sum_{k=0}^{n}(-1)^k G_{k+3}G_k = \frac{1}{1278}((-1)^n(-5G_{n+4}^2 - 379G_{n+3}^2 - 140G_{n+2}^2 + 245G_{n+1}^2 + 202G_{n+4}G_{n+3} + 188G_{n+4}G_{n+2} + 4G_{n+4}G_{n+1} - 1124G_{n+3}G_{n+2} - 1106G_{n+3}G_{n+1} + 322G_{n+2}G_{n+1}) + 5)$.

Taking $W_n = H_n$ with $H_0 = 4, H_1 = 2, H_2 = 10, H_3 = 41$ in the last proposition, we have the following corollary which presents sum formulas of Lucas 4-primes numbers.

Corollary 4.15. *For $n \geq 0$, Lucas 4-primes numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k H_k^2 = \frac{1}{1278}((-1)^n(13H_{n+4}^2 - 37H_{n+3}^2 + 364H_{n+2}^2 + 641H_{n+1}^2 - 14H_{n+4}H_{n+3} - 148H_{n+4}H_{n+2} - 266H_{n+4}H_{n+1} + 196H_{n+3}H_{n+2} + 490H_{n+3}H_{n+1} + 952H_{n+2}H_{n+1}) - 499)$.

(b) $\sum_{k=0}^{n}(-1)^k H_{k+1}H_k = \frac{1}{1278}((-1)^n(19H_{n+4}^2 + 77H_{n+3}^2 + 106H_{n+2}^2 - 931H_{n+1}^2 - 86H_{n+4}H_{n+3} - 118H_{n+4}H_{n+2} + 70H_{n+4}H_{n+1} + 352H_{n+3}H_{n+2} + 28H_{n+3}H_{n+1} - 542H_{n+2}H_{n+1}) - 4465)$.

(c) $\sum_{k=0}^{n}(-1)^k H_{k+2}H_k = \frac{1}{426}((-1)^n(H_{n+4}^2 + 19H_{n+3}^2 - 256H_{n+2}^2 - 49H_{n+1}^2 - 12H_{n+4}H_{n+3} + 76H_{n+4}H_{n+2} + 56H_{n+4}H_{n+1} - 116H_{n+3}H_{n+2} - 148H_{n+3}H_{n+1} - 462H_{n+2}H_{n+1}) + 333)$.

(d) $\sum_{k=0}^{n}(-1)^k H_{k+3}H_k = \frac{1}{1278}((-1)^n(-5H_{n+4}^2 - 379H_{n+3}^2 - 140H_{n+2}^2 + 245H_{n+1}^2 + 202H_{n+4}H_{n+3} + 188H_{n+4}H_{n+2} + 4H_{n+4}H_{n+1} - 1124H_{n+3}H_{n+2} - 1106H_{n+3}H_{n+1} + 322H_{n+2}H_{n+1}) - 8197)$.

From the last proposition, we have the following corollary which gives sum formulas of modified 4-primes numbers (take $W_n = E_n$ with $E_0 = 0, E_1 = 0, E_2 = 1, E_3 = 1$).

Corollary 4.16. *For $n \geq 0$, modified 4-primes numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k E_k^2 = \frac{1}{1278}((-1)^n (13E_{n+4}^2 - 37E_{n+3}^2 + 364E_{n+2}^2 + 641E_{n+1}^2 - 14E_{n+4}E_{n+3} - 148E_{n+4}E_{n+2} - 266E_{n+4}E_{n+1} + 196E_{n+3}E_{n+2} + 490E_{n+3}E_{n+1} + 952E_{n+2}E_{n+1}) - 38)$.

(b) $\sum_{k=0}^{n}(-1)^k E_{k+1}E_k = \frac{1}{1278}((-1)^n (19E_{n+4}^2 + 77E_{n+3}^2 + 106E_{n+2}^2 - 931E_{n+1}^2 - 86E_{n+4}E_{n+3} - 118E_{n+4}E_{n+2} + 70E_{n+4}E_{n+1} + 352E_{n+3}E_{n+2} + 28E_{n+3}E_{n+1} - 542E_{n+2}E_{n+1}) + 10)$.

(c) $\sum_{k=0}^{n}(-1)^k E_{k+2}E_k = \frac{1}{426}((-1)^n (E_{n+4}^2 + 19E_{n+3}^2 - 256E_{n+2}^2 - 49E_{n+1}^2 - 12E_{n+4}E_{n+3} + 76E_{n+4}E_{n+2} + 56E_{n+4}E_{n+1} - 116E_{n+3}E_{n+2} - 148E_{n+3}E_{n+1} - 462E_{n+2}E_{n+1}) + 8)$.

(d) $\sum_{k=0}^{n}(-1)^k E_{k+3}E_k = \frac{1}{1278}((-1)^n (-5E_{n+4}^2 - 379E_{n+3}^2 - 140E_{n+2}^2 + 245E_{n+1}^2 + 202E_{n+4}E_{n+3} + 188E_{n+4}E_{n+2} + 4E_{n+4}E_{n+1} - 1124E_{n+3}E_{n+2} - 1106E_{n+3}E_{n+1} + 322E_{n+2}E_{n+1}) - 182)$.

4.3 The case $x = i$

In this subsection we consider the special case $x = i$. Taking $r = s = t = u = 1$ in Theorem 3.1, we obtain the following proposition.

Proposition 4.4. *If $r = s = t = u = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} i^k W_k^2 = \frac{1}{5-4i}(i^n (iW_{n+4}^2 + 4iW_{n+3}^2 + (3-2i)W_{n+2}^2 - (5+5i)W_{n+1}^2 + (1-5i)W_{n+4}W_{n+3} + (1+i)W_{n+4}W_{n+2} + (3+i)W_{n+4}W_{n+1} - (7-i)W_{n+3}W_{n+2} - (3-i)W_{n+3}W_{n+1} - (3-3i)W_{n+2}W_{n+1}) - W_3^2 - 4W_2^2 + (2+3i)W_1^2 + (5-5i)W_0^2 + (5+i)W_3W_2 - (1-i)W_3W_1 - (1-3i)W_3W_0 - (1+7i)W_2W_1 - (1+3i)W_2W_0 - (3+3i)W_1W_0)$.

(b) $\sum_{k=0}^{n} i^k W_{k+1}W_k = \frac{1}{10-8i}(i^n ((1-3i)W_{n+4}^2 - (6+4i)W_{n+3}^2 - (2-4i)W_{n+2}^2 + (3+i)W_{n+1}^2 + (2+6i)W_{n+4}W_{n+3} - (3-5i)W_{n+4}W_{n+2} + 4iW_{n+4}W_{n+1} + (5-7i)W_{n+3}W_{n+2} + (1-i)W_{n+3}W_{n+1} - (6+8i)W_{n+2}W_{n+1}) + (3+i)W_3^2 + (4-6i)W_2^2 - (4+2i)W_1^2 - (1-3i)W_0^2 - (6-2i)W_3W_2 - (5+3i)W_3W_1 - 4W_3W_0 + (7+5i)W_2W_1 + (1+i)W_2W_0 + (8-6i)W_1W_0)$.

(c) $\sum_{k=0}^{n} i^k W_{k+2}W_k = \frac{1}{10-8i}(i^n (2W_{n+4}^2 - (1+i)W_{n+3}^2 - (5-3i)W_{n+2}^2 + 2iW_{n+1}^2 - (1+i)W_{n+4}W_{n+3} + (2-2i)W_{n+4}W_{n+2} + (1+3i)W_{n+4}W_{n+1} - (6-4i)W_{n+3}W_{n+2} - (6+4i)W_{n+3}W_{n+1} - (3-5i)W_{n+2}W_{n+1}) + 2iW_3^2 + (1-i)W_2^2 - (3+5i)W_1^2 - 2W_0^2 - (5+3i)W_1W_0 + (4-6i)W_2W_0 - (3-i)W_3W_0 - (4+6i)W_2W_1 + (2+2i)W_3W_1 + (1-i)W_3W_2)$.

(d) $\sum_{k=0}^{n} i^k W_{k+3}W_k = \frac{1}{10-8i}(i^n (6W_{n+4}^2 - (3+3i)W_{n+3}^2 - (5-i)W_{n+2}^2 + 6iW_{n+1}^2 - (3+3i)W_{n+4}W_{n+3} - (4-2i)W_{n+4}W_{n+2} - (5+i)W_{n+4}W_{n+1} - (8-4i)W_{n+3}W_{n+2} - (2-8i)W_{n+3}W_{n+1} + (1+7i)W_{n+2}W_{n+1}) + 6iW_3^2 + (3-3i)W_2^2 - (1+5i)W_1^2 - 6W_0^2 + (3-3i)W_3W_2 - (2+4i)W_3W_1 + (1-5i)W_3W_0 - (4+8i)W_2W_1 - (8+2i)W_2W_0 - (7-i)W_1W_0)$.

From the above proposition, we have the following corollary which gives sum formulas of Tetranacci numbers (take $W_n = M_n$ with $M_0 = 0, M_1 = 1, M_2 = 1, M_3 = 2$).

Corollary 4.17. *For $n \geq 0$, Tetranacci numbers have the following properties:*

(a) $\sum_{k=0}^{n} i^k M_k^2 = \frac{1}{5-4i}(i^n (iM_{n+4}^2 + 4iM_{n+3}^2 + (3-2i)M_{n+2}^2 - (5+5i)M_{n+1}^2 + (1-5i)M_{n+4}M_{n+3} + (1+i)M_{n+4}M_{n+2} + (3+i)M_{n+4}M_{n+1} - (7-i)M_{n+3}M_{n+2} - (3-i)M_{n+3}M_{n+1} - (3-3i)M_{n+2}M_{n+1}) + 1)$.

(b) $\sum_{k=0}^{n} i^k M_{k+1}M_k = \frac{1}{10-8i}(i^n ((1-3i)M_{n+4}^2 - (6+4i)M_{n+3}^2 - (2-4i)M_{n+2}^2 + (3+i)M_{n+1}^2 + (2+6i)M_{n+4}M_{n+3} - (3-5i)M_{n+4}M_{n+2} + 4iM_{n+4}M_{n+1} + (5-7i)M_{n+3}M_{n+2} + (1-i)M_{n+3}M_{n+1} - (6+8i)M_{n+2}M_{n+1}) - 3 - i)$.

(c) $\sum_{k=0}^{n} i^k M_{k+2}M_k = \frac{1}{10-8i}(i^n (2M_{n+4}^2 - (1+i)M_{n+3}^2 - (5-3i)M_{n+2}^2 + 2iM_{n+1}^2 - (1+i)M_{n+4}M_{n+3} + (2-2i)M_{n+4}M_{n+2} + (1+3i)M_{n+4}M_{n+1} - (6-4i)M_{n+3}M_{n+2} - (6+4i)M_{n+3}M_{n+1} - (3-5i)M_{n+2}M_{n+1}) - 2i)$.

(d) $\sum_{k=0}^{n} i^k M_{k+3}M_k = \frac{1}{10-8i}(i^n (6M_{n+4}^2 - (3+3i)M_{n+3}^2 - (5-i)M_{n+2}^2 + 6iM_{n+1}^2 - (3+3i)M_{n+4}M_{n+3} - (4-2i)M_{n+4}M_{n+2} - (5+i)M_{n+4}M_{n+1} - (8-4i)M_{n+3}M_{n+2} - (2-8i)M_{n+3}M_{n+1} + (1+7i)M_{n+2}M_{n+1}) - 6i)$.

Taking $W_n = R_n$ with $R_0 = 4, R_1 = 1, R_2 = 3, R_3 = 7$ in the above proposition, we have the following corollary which presents sum formulas of Tetranacci-Lucas numbers.

Corollary 4.18. *For $n \geq 0$, Tetranacci-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} i^k R_k^2 = \frac{1}{5-4i}(i^n (iR_{n+4}^2 + 4iR_{n+3}^2 + (3-2i)R_{n+2}^2 - (5+5i)R_{n+1}^2 + (1-5i)R_{n+4}R_{n+3} + (1+i)R_{n+4}R_{n+2} + (3+i)R_{n+4}R_{n+1} - (7-i)R_{n+3}R_{n+2} - (3-i)R_{n+3}R_{n+1} - (3-3i)R_{n+2}R_{n+1}) + 40 - 34i)$.

(b) $\sum_{k=0}^{n} i^k R_{k+1}R_k = \frac{1}{10-8i}(i^n((1-3i)R_{n+4}^2-(6+4i)R_{n+3}^2-(2-4i)R_{n+2}^2+(3+i)R_{n+1}^2+(2+6i)R_{n+4}R_{n+3}-(3-5i)R_{n+4}R_{n+2}+4iR_{n+4}R_{n+1}+(5-7i)R_{n+3}R_{n+2}+(1-i)R_{n+3}R_{n+1}-(6+8i)R_{n+2}R_{n+1})-45+65i)$.

(c) $\sum_{k=0}^{n} i^k R_{k+2}R_k = \frac{1}{10-8i}(i^n(2R_{n+4}^2-(1+i)R_{n+3}^2-(5-3i)R_{n+2}^2+2iR_{n+1}^2-(1+i)R_{n+4}R_{n+3}+(2-2i)R_{n+4}R_{n+2}+(1+3i)R_{n+4}R_{n+1}-(6-4i)R_{n+3}R_{n+2}-(6+4i)R_{n+3}R_{n+1}-(3-5i)R_{n+2}R_{n+1})-59+3i)$.

(d) $\sum_{k=0}^{n} i^k R_{k+3}R_k = \frac{1}{10-8i}(i^n(6R_{n+4}^2-(3+3i)R_{n+3}^2-(5-i)R_{n+2}^2+6iR_{n+1}^2-(3+3i)R_{n+4}R_{n+3}-(4-2i)R_{n+4}R_{n+2}-(5+i)R_{n+4}R_{n+1}-(8-4i)R_{n+3}R_{n+2}-(2-8i)R_{n+3}R_{n+1}+(1+7i)R_{n+2}R_{n+1})-129-13i)$.

Corresponding sums of the other fourth order generalized Tetranacci numbers can be calculated similarly.

5 Numerical Examples

In this section, for the specific cases of x and n, we present numerical computations of sums $\sum_{k=0}^{n} x^k W_k^2$, $\sum_{k=0}^{n} x^k W_{k+1}W_k$, $\sum_{k=0}^{n} x^k W_{k+2}W_k$ and $\sum_{k=0}^{n} x^k W_{k+3}W_k$ for the specific case of sequence $\{W_n\}$. We only consider Tetranacci and Tetranacci-Lucas cases. The other corresponding numerical sums of the other fourth order generalized Tetranacci numbers can be calculated similarly.

5.1 The case $x = 1$

In this subsection we consider the special case $x = 1$ and $n = 9$.

Taking $r = s = t = u = 1, x = 1, n = 9$ in Theorem 3.1, we have the following corollary which gives sum formulas of Tetranacci numbers (take $W_n = M_n$ with $M_0 = 0, M_1 = 1, M_2 = 1, M_3 = 2$).

Corollary 5.1. *Tetranacci numbers have the following properties:*

(a) $\sum_{k=0}^{9} M_k^2 = 15952$.

(b) $\sum_{k=0}^{9} M_{k+1}M_k = 30734$.

(c) $\sum_{k=0}^{9} M_{k+2}M_k = 59242$.

(d) $\sum_{k=0}^{9} M_{k+3}M_k = 114198$.

Taking $W_n = R_n$ with $R_0 = 4, R_1 = 1, R_2 = 3, R_3 = 7$ and $x = 1, n = 9$ in Theorem 3.1, we have the following corollary which presents sum formulas of Tetranacci-Lucas numbers.

Corollary 5.2. *Tetranacci-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{9} R_k^2 = 184548$.

(b) $\sum_{k=0}^{9} R_{k+1}R_k = 355740$.

(c) $\sum_{k=0}^{9} R_{k+2}R_k = 685842$.

(d) $\sum_{k=0}^{9} R_{k+3}R_k = 1322030$.

5.2 The case $x = -1$

In this subsection we consider the special case $x = -1$ and $n = 9$.

Taking $r = s = t = u = 1, x = -1, n = 9$ in Theorem 3.1, we have the following corollary which gives sum formulas of Tetranacci numbers (take $W_n = M_n$ with $M_0 = 0, M_1 = 1, M_2 = 1, M_3 = 2$).

Corollary 5.3. *Tetranacci numbers have the following properties:*

(a) $\sum_{k=0}^{9}(-1)^k M_k^2 = -9196$.

(b) $\sum_{k=0}^{9}(-1)^k M_{k+1}M_k = -17700$.

(c) $\sum_{k=0}^{9}(-1)^k M_{k+2}M_k = -34138.$

(d) $\sum_{k=0}^{9}(-1)^k M_{k+3}M_k = -65798.$

Taking $W_n = R_n$ with $R_0 = 4, R_1 = 1, R_2 = 3, R_3 = 7$ and $x = -1, n = 9$ in Theorem 3.1, we have the following corollary which presents sum formulas of Tetranacci-Lucas numbers.

Corollary 5.4. *Tetranacci-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{9}(-1)^k R_k^2 = -105884.$

(b) $\sum_{k=0}^{9}(-1)^k R_{k+1}R_k = -204618.$

(c) $\sum_{k=0}^{9}(-1)^k R_{k+2}R_k = -394260.$

(d) $\sum_{k=0}^{9}(-1)^k R_{k+3}R_k = -759984.$

5.3 The case $x = i$

In this subsection we consider the special case $x = i$ and $n = 9$.

Taking $r = s = t = u = 1, x = i, n = 9$ in Theorem 3.1, we have the following corollary which gives sum formulas of Tetranacci numbers (take $W_n = M_n$ with $M_0 = 0, M_1 = 1, M_2 = 1, M_3 = 2$).

Corollary 5.5. *Tetranacci numbers have the following properties:*

(a) $\sum_{k=0}^{9} i^k M_k^2 = 2926 + 10884i.$

(b) $\sum_{k=0}^{9} i^k M_{k+1}M_k = 5643 + 20953i.$

(c) $\sum_{k=0}^{9} i^k M_{k+2}M_k = 10864 + 40394i.$

(d) $\sum_{k=0}^{9} i^k M_{k+3}M_k = 20944 + 77874i.$

Taking $W_n = R_n$ with $R_0 = 4, R_1 = 1, R_2 = 3, R_3 = 7$ and $x = i, n = 9$ in Theorem 3.1, we have the following corollary which presents sum formulas of Tetranacci-Lucas numbers.

Corollary 5.6. *Tetranacci-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{9} i^k R_k^2 = 34112 + 125516i.$

(b) $\sum_{k=0}^{9} i^k R_{k+1}R_k = 65421 + 242151i.$

(c) $\sum_{k=0}^{9} i^k R_{k+2}R_k = 126219 + 467021i.$

(d) $\sum_{k=0}^{9} i^k R_{k+3}R_k = 243433 + 900109i.$

6 Conclusion

Recently, there have been so many studies of the sequences of numbers in the literature and the sequences of numbers were widely used in many research areas, such as architecture, nature, art, physics and engineering. In this work, sum identities were proved. The method used in this paper can be used for the other linear recurrence sequences, too. We have written sum identities in terms of the generalized tetranacci sequence, and then we have presented the formulas as special cases the corresponding identity for the Tetranacci, Tetranacci-Lucas and some other fourth order linear recurrance sequences. All the listed identities in the corollaries may be proved by induction, but that method of proof gives no clue about their discovery. We give the proofs to indicate how these identities, in general, were discovered.

Computations of the Frobenius norm, spectral norm, maximum column length norm and maximum row length norm of circulant (r-circulant, geometric circulant, semicirculant) matrices with the generalized m-step Fibonacci sequences require the sum of the squares of the numbers of the sequences.

Competing Interests

Author has declared that no competing interests exist.

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