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Some Topological and Algebraic Features of Symmetric Spaces

Um Salama[1*] Ahmed Abd Alla[1] and A. Elemam[2]

[1]Department of Mathematics, Faculty of Darb College University, Saudi Arabia.
[2]Department of Mathematics, Faculty of Education Jazan University, Saudi Arabia.

Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

In this study, we introduce some approaches, geometrical and algebraic, which help to give further understanding of symmetric spaces. Symmetric space is a very important field for understanding abstract and applied features of spaces. We have introduced Riemannian Manifold, Lie groups and Lie algebras, and some of their topological and algebraic properties, with some concentration on Lie algebras and root systems , which help classification and many applications of symmetric spaces. The paper is an attempt to explain some algebraic features of symmetric spaces and how to get some of their properties using algebraic approach, concluded with some results.

Keywords: Topological spaces; metric spaces; topological manifold; Riemannian manifold; lie groups; lie algebras; root systems; homogeneous spaces; symmetric spaces.

1 Introduction

In studying spaces, one of the aims of this study is to introduce spaces that can suit some scientific application. Many scientific problems in various fields may have their own conditions that might not agree with the geometric structure and properties of some spaces familiar to mathematicians and

**Corresponding author: E-mail: selma480@hotmail.com;*

geometers. Many properties of symmetric spaces can be studied through their Lie algebras and root systems, and specially the problem of classification of symmetric spaces [1,2].

Various applications of Lie algebras and symmetric spaces in different fields, especially in physics. In mathematical context, in this paper we are treating some algebraic and topological properties of Lie algebras associated to symmetric spaces to make it possible for further understanding and carrying more applications [3-7].

2 Topological Spaces

Toplogical spaces are mathematical structures that allow the formal definition of concepts such as convergence, connectedness and continuity. They appear virtually in every branch of modern mathematic and are central unifying notions.

The branch of mathematics that studies toplogical spaces in their own right is called topology.

Definition of topological spaces 2.1

Let X be a set, let T be a collection of subsets such that

1. The union of a family of sets which are elements of T belongs to T.
2. The intersection of a finite family of sets which are elements of T belongs to T.
3. The empty set$\emptyset$ and the whole X belong to T. Then

- T is called a topological structure or just a topology in X
- The pair (X,T) is called a topological space.
- The element of X is called point of this topological space.
- The element of T is called open set of the topological space (X,T) .The conditions in the definition above are called the axioms of topological structure.

Examples 2.2

1. A discrete topological space is a set with the topological structure which consists of all the subsets.
2. The Euclidean spaces R^n can be given a topology in the usual topology onR^n , the basic open sets are the open balls.

3 Metric Spaces

Definition 3.1

A metric space is a set with a function that satisfies.

$d: X \times X \to R_+$ that satisfies.

1. $d(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$
2. $d(x,y) = d(y,x) = 0$ for every $x, y \in X$
3. $d(x,y) + d(y,z) \geq d(x,z)$ triangle inequality

The pair (X, d) is called metric space.

Example 3.2

The usual metric on C (complex numbers) is the Euclidean metric determined by the modulus function $((\mathrm{z},\mathrm{w})) \rightarrow |\mathrm{z}-\mathrm{w}|$. It is of course an extension to $G \times G$ of the Euclidean metric on R. We shall assume that G is endowed with it unless we state otherwise.

Theorem 3.3 [8]

Suppose(X, d) is a metric space. The function is objective function from X on to $d(X)$.

Theorem 3.4 [8]

Suppose X is a metric space, Z is a metric subspace of X and S⊆Z. Then S is a connected subset of X if and only if S is a connected subset of Z.

4 Topological Manifold

Euclidean space and their subspace R^n are the most important .The metric space R^n serve as a topological model for Euclidean space E^n**,** for finite dimensional vector spaces over R or C . It is natural enough that we are led to study those spaces which are locally like R^n . We will consider spaces called manifolds**,** defined as follows.

Definition 4.1

A manifold M of dimension n , or n-manifold is topological space with the following properties:

I. M is Housdorff space.
II. M is locally Euclidean of dimension n and,
III. M has a countable basis of open sets .

As a matter of notion dim M is used for the dimension of M , when

dim $=0$, then M is a countable space with discrete topology

Example 4.2

Define the circle $S^1 = \{z \in \mathbb{C}: |z| = 1\}$. Then for any fixed point $z \in S^1$,write it as $z = e^{2\pi i c}$ for a unique real number $0 \leq c \leq 1$, and define the map

$$v_z: t \rightarrow e^{2\pi i t} .$$

We note that v_z maps the natural $\mathrm{I_c} = (\mathrm{c}-\frac{1}{2}, \mathrm{c}+\frac{1}{2})$ to the neighborhood of z given by $\mathrm{s}^1/{-\mathrm{z}}$, and it is a homeomorphism. Then $\varphi_z = v_z|_{\mathrm{I_c}}^{-1}$ is a local coordinate chart near. By taking products of coordinate charts, we obtain charts for the Cartesian product of manifolds. Hence the Cartesian product is a manifold

Theorem 4.3 [8]

A topological manifold M is locally connected, locally compact, and a union of a countable collection of compact subsets; furthermore, it is normal and metrizable.

5 Riemannian Manifold

In this section we introduce the notion of a Riemannian manifold $(M.g)$.The metric g provides us with an inner product on each tangent space an can be used to measure angels and the lengths of curve in the manifold .These terms are named after the German mathematician Bernhard Riemann.

This defines a distance function and turns the manifold into a metric space in a natural way.

Let M be a smooth manifold, $C^{\infty}(M)$ denote the commutative ring of smooth function on M and $C^{\infty}(TM)$ be the set of smooth vector fields on M forming a module over $C^{\infty}(M)$ .Put $C_0^{\infty}(TM) = C^{\infty}(M)$ and for each positive integer

$r \in Z^{+}$ let $C_r^{\infty}(TM) = C^{\infty}(M) \otimes \ldots \otimes \ldots \otimes C^{\infty}(TM)$ be the r-fold tensor product of $C^{\infty}(TM)$ over $C^{\infty}(M)$.

Definition 5.1

a. A Riemannian manifold is a pair $(M.g)$ consisting of a smooth manifold M and a metric g on the tangent bundle, i.e., a smooth, symmetric positive definite $(0,2)$ -tensor field on M .The tensor g is called a Riemannian metric on .
b. Two Riemannian manifolds $M_i, M_i)$ $(i = 1,2)$ are said to be isometric if there exists a diffeomorphism $\Phi : M_1 \to M_2$ such that $\Phi^* g_2 = g_1$.

Examples 5.2

(1) (The Euclidean space): The space R^n has a natural metric

$$g_0 = (dx^1)^2 + \cdots + (dx^n)^2.$$

The geometry of (R^n, g_0) is the classical Euclidean geometry.

(2) (The hyperbolic plane): The Poincare model of the hyperbolic plan is the Riemannian manifold (D, g) where D is the unit open disk in the plane R^2 and the metric g is given by

$$g = \frac{1}{1 - x^2 - y^2}(dx^2 + dy^2).$$

Theorem 5.3 (Fundental Theorem of Riemannian Geometry)

Let M be a Riemannian manifold, there exists a uniquely determined Riemannian connection on M.

Theorem 5.4 [8]

A connected Riemannian manifold is a metric space with the metric $d(p, q) = infimum$ of the lengths of curves of class C^1 from p to q its metric space topology and manifold topology agree.

6 Lie Groups

Definition 6.1

A Lie group G is a group satisfying the well-known axioms of group, besides the mappings $G \times G \to G$ *and* $C \to G^{-1}$ defined by $(x, y) \to xy$ and $x \to x^{-1}$

Respectively are both C^{∞}. This definition implies that the Lie group G is a differentiable manifold. Lie groups are very important due to the fact that, their algebraic properties derive from group axioms, and their geometric properties derive from the identification of group operations with points in a topological space.

Examples 6.2

i. The set C^* of nonzero complex numbers is a 2-dimensional Lie group under complex multiplication which can be identified with $Gl(1, C)$.
ii. The set $GL(n, R)$ of nonsingular $n \times n$ matrices is a group with respect to matrix multiplication. . An $n \times n$ matrix X is nonsingular if and only if $\det X \neq 0$. If $X, Y \in GL(n, R)$ then both the maps$(X, Y) \to XY$ and $x \to x^{-1}$ are C^{∞}. Thus $GL(n, R)$ is a Lie group.
iii. The Euclidean space R^n under addition is a group endowed with the smooth operations $(x, y) \to x + y$ and $x \to x^{-1}$ $\forall$ $x, y \in R^n$ forms a Lie group.

There are many other examples for Lie groups and their applications which can be seen in various references. The matrices in $GL(n, R)$ can be represented as

$$M = exp(\sum_i t^i X_i) \tag{6.1}$$

Where X_i are the generators of what is called the Lie algebra of the Lie group and t^i are real parameters. For a Lie group the tangent space at the origin is spanned by the generators, considered as vector fields which are expressed as

$X = X^i(x)\frac{\partial}{\partial x^i}$, where the partial derivatives $\frac{\partial}{\partial x^i}$ form a basis for the vector field. If X is a generator of a lie group G , then X onto exp^{tX} is the exponential map , which is a one - parameter subgroup, defining a curve $c(t)$ in the group manifold. For the curve $c(t)$ the tangent vector at the origin is given by

$$\frac{d}{dt}e^{tX}| \, t \, = X \tag{6.2}$$

The matrix exponential is very useful because it is always nonsingular since

$\det(e^x)=$ e^{tX} is never zero.

7. The Lie Algebra

In this section, we review basic concepts of Lie algebras, besides some of their properties needed in studying symmetric spaces. To study a Lie algebra, we must know that it is a linearization of its original Lie group, so one remembers that a Lie group is a group provided that its two operations: multiplication and inversion are smooth maps.

Definition 7.1

A Lie algebra is a pair $(V, [\,,])$ where V is a vector space and [,] is a Lie bracket, $[\ ,\ ]: V \times V \to V$ satisfying:

(1) $[v, w] = -[w, v]$ skew-symmetric.
(2) $[av + bu, w] = a[v, w] + b[u, w]$ a bilinear .
(3) $[v, [w, u]] + [w, [u, v]] + [u, [v, w]] = 0$

For all, u and $w \in V$. a Bianchi identity.

A Lie Bracket is a binary operation [,] on a vector space V

Example 7.2

Let $V = R^3$, [,]: $R^3 \times R^3 \rightarrow R^3$ as proved that it is a Lie algebra.

Example 7.3

Let $\Omega(M)$ be the set of all vector fields on a manifold M .

Define $[\, v , w \,] = vw - wv$,

Then $[v, w]$ is a Lie bracket.

A homeomorphism of Lie algebra ℓ is a linear map, : $\ell \rightarrow \acute{\ell}$, preserving the bracket . This means that

$\varphi[\ell_1, \ell_2] = [\varphi(\ell_1), \varphi(\ell_2)]$ for any $(\ell_1, \ell_2) \in \ell \times \ell$.

A Lie sub algebra of Lie algebra ℓ is a sub-vector space η such that

$[\eta, \eta] \subseteq \eta$. An ideal of ℓ is a Lie subalgebra η such that $[\eta, \ell] \subseteq \eta$

A vector subspace η of a Lie algebra ℓ is called a Lie sub algebra if $[\eta, \ell] \subseteq \eta$.

Theorem 7.4 [9]

Let G be a Lie group and ℓ its Lie algebra:

(1) If H is a Lie subgroup of G, η is a Lie subalgebra of ℓ.
(2) If η is a Lie subalgebra, there exists a unique Lie subgroup H of G such that Lie algebra of H is isomorphic to η

A Lie algebra is an algebraic structure whose main use is in studying geometric objects such as Lie groups and differentiable manifolds. Every Lie group G has a corresponding Lie algebra denoted by g , it is the tangent space at the identity of the Lie group G . The Lie algebra generates a group through the exponential mapping.

Example 7.5

The Lie algebra g of R^n as a Lie group is again where $[X, Y] = 0 \;\; \forall x, y \in g$

Thus the Lie bracket for the Lie algebra of any abelian group is zero.

Definition 7.6 (Ideals)

An ideal $\boldsymbol{I}$ of a Lie algebra g is a sub algebra such that $[\mathbf{g} , \mathbf{I}] \subset \mathbf{I}$, also an abelian ideal satisfies $[\mathbf{I} , \mathbf{I}] = 0$

Ideals in Lie algebras perform like normal subgroups in group theory, they can be used in analyzing the structure of Lie algebras and in constructing quotient algebras. Also a Lie algebra is abelian if and only if its center $\mathrm{Z}(g) = g$ this is because the center $\mathrm{Z}(g) = \{\mathrm{Z} \in g \mid [\mathrm{X}, \mathrm{Z}] = 0 \;, \forall x , \mathrm{Z} \in g\}$ is also an ideal of g, and g is abelian if $[X , Y] = 0$, $\forall X , \mathrm{Y} \in g$.

Definition 7.7 (Simple and semi simple Lie algebras)

A simple Lie algebra $\boldsymbol{g}$ has no proper ideals or in other words, a simple Lie algebra has no ideals except itself and **0** and $[g, g_{,}] \neq 0$ **.**

A semisimple Lie algebra is the direct sum of simple algebras , and has no proper abelian ideal. If $\boldsymbol{g}$ is simple then Z $(g\ \boldsymbol{)} = \boldsymbol{0}$ and $[g, g_{,}] = g$. When a Lie algebra g is not simple, we can factor out a nonzero proper ideal $\boldsymbol{h}$ to get a Lie algebra of smaller dimension, which we call it a **quotient algebra**, denoted by $^{g}/_{h}$.

Definition 7.8 (Derived algebra)

It is the collection of all linear combinations of $[X\ ,\ Y\ ]\ \forall X, Y \in g$ and it is denoted by $[\boldsymbol{g}, \boldsymbol{g}]$**.** It is also an ideal and determines whether the Lie algebra is abelian or not, in fact we can say that the Lie algebra g is abelian if and only if its derived algebra is the zero vector.

Definition 7.9 (Solvable Lie algebra)

A Lie algebra g is solvable if its derived series goes down to zero that is $g^{(n)} = 0$ for some $n \in N$

We remark that any abelian Lie algebra is solvable and any simple algebra is nonsalable .The following proposition gives some facts about solvability:

Proposition 7.10 [10]

- i Given $\boldsymbol{g}$ is a solvable Lie algebra, then all sub algebras and homomorphic images of g are also solvable.
- ii If $\boldsymbol{h}$ is a solvable ideal of a Lie algebra $\boldsymbol{g}$ such that $^{\boldsymbol{g}}/_{\boldsymbol{h}}$ is solvable, then **g** is solvable as well.
- iii Suppose $\boldsymbol{h}$ and r are solvable ideals of a Lie algebra $\boldsymbol{g}$, then $\boldsymbol{h} + \boldsymbol{r}$ is solvable.

Definition 7.11 (The radical)

In the Lie algebra $\boldsymbol{g}$, the unique maximal solvable ideal is called the **radical** of $\boldsymbol{g}$ denoted rad $\boldsymbol{g}$**.** Suppose $\boldsymbol{g}$ is an arbitrary Lie algebra , $\boldsymbol{r}$ is an ideal included in no larger solvable ideal , and any $\boldsymbol{h}$ other solvable ideal of $\boldsymbol{g}$. Using maximality and Prop.3.10 we have $\boldsymbol{h} + \boldsymbol{r} = \boldsymbol{r}$, which means $\boldsymbol{h} \subset \boldsymbol{r}$ **h** and $\boldsymbol{r}$ is unique. It can be shown that a Lie algebra $\boldsymbol{g}$ is semisimple if rad $\boldsymbol{g}$ = **0** and a simple algebra $\boldsymbol{g}$ is also semisimple but the converse is not true.

Definition 7.12 (Nilpotent Lie algebras)

A Lie algebra g is called nilpotent if g $^{(n)} = 0$ for some $n \in \mathbb{N}$ where g $^{(n)}$ is an element the descending central series written as

$$g^0 = g,\ g^1 = [g, g]\ ,\ g^2 = [g,\ g^1]\ , \dots \dots , g^i = [g, g^{i-1}].$$

Any Abelian Lie algebra is nilpotent since $g^1 = [g, g] = 0$ and all nilpotent Lie algebras are solvable [9].

Definition 7.13 (Nilpotent Endomorphism)

An adjoint representation $ad\ x_i$ as an endomorphism in the Lie algebra **g** is a nilpotent endomorphism if $ad\ x$ $ad\ x_{1\ 2} \dots ad\ x_n (y) = 0\ \forall x_i, y \in$ **g .** Or in another way $[x_1,[x_2,[\dots[x_n, y]\dots]=0$.

We say that an element $x \in \mathbf{g}$ is **ad-nilpotent** if it has a nilpotent endomorphism so if **g** is nilpotent , then all of its elements are adnilpotent.

Engel's Theorem 7.14 [9]

If all elements of **g** are **ad-nilpotent**, then **g** is nilpotent.

This theorem is very important because it helps us to show that a Lie algebra is nilpotent without directly calculating its descending **central series.**

Proposition 7.15 [10]

i) Suppose $\mathbf{g}$ is a nilpotent Lie algebra . Then all subalgebras and homomorphic images of **g** are also nilpotent.
ii) If **g** / Z(**g**) is nilpotent , then **g** is nilpotent as well.
iii) If **g** is nilpotent and nonzero, then Z ($\mathbf{g}$) $\neq \mathbf{0}$.

Definition 7.16 (The killing form)

It is the symmetric bilinear denoted k (x y,) and can be found using the adjoint representation, where $_k(x\ y,)$ = tr (adx, ady) for x y, ∈ **g.**

A killing form is nondegenerate if its radical **r** = 0, where **r** = {x ∈ **g** | $_k(x\ y,) = 0\ \forall\ y \in g\}$. A nondegenerate Killing form gives useful information as for example in Cartan's first and second criteria. Also we have Weyl theorem [10] which states that a simple Lie algebra **g** is compact ,if and only if the killing form on **g** is negative definite, otherwise it is noncompact and this very useful in studying symmetric spaces. Also to determine the solvability of a Lie algebra we can use Cartan's first criterion as follows:

Theorem 7.17 [9] (Cartan's first criterion)

A Lie algebra **g** is solvable if and only if $_k(x\ y,) = 0$ for all x ∈**[g**, **g**], y ∈ **g**.

We use Cartan's second criterion to determine if a Lie algebra is semisimple or not as follows:

Theorem 7.18 [9] (Cartan's second criterion)

A Lie algebra **g** is semisimple if and only if its killing form is nondegenerate.

When we study the structure of a Lie algebra , its solvability and simplicity are helpful in this field , and also when we can decompose the Lie algebra into simple ideals or using its semi simplicity.

8 Root Systems

Before we introduce root systems of Lie algebras we give some preliminary notions which help in understanding the required ideas. Also it is worth mentioning that root systems are very effective tools which are used in classifying and studying the structure of Lie algebras.

Cartan subalgebras 8.1

A cartan sub algebra of g is sub algebra $\mathfrak{h}$ $of\ g$ satisfying the following condition

(i) $\mathfrak{h}$ is a maximal a belian sub algebra of g
(ii) for each $H \in \mathfrak{h}$ the endomorphism ad $H\ of\ g$ is semisimple

Definition:

The element $H \in g$ is called regular if

$\dim g(\mathrm{H}, \mathrm{o}) = \min\left(\dim g\ (\mathrm{x}, \mathrm{o})\right),\ \mathrm{x} \in g$.

Theorem 8.2 [9]

Let H_0 be a regular element in g $.Then$ $g(H_0, 0)$ is a cartan subalgebra of g.

Lemma 8.3 [9]

The algebra $\mathfrak{h}$ is abelian and infact a maximal a belian sub algebra of g.

Definition 8.4

Let g be lie algebra. Alin subalgebra $\mathfrak{h}$ of g is a cartan sub algebra of g if

(i) $\mathfrak{h}$ is enilpotent lie algebra
(ii) $\mathfrak{h}$ is equal to it's own normalizer

9 Symmetric Spaces

Symmetric spaces are of great importance for several branches of mathematics. Any symmetric space has its own special geometry, such as Euclidean, elliptic & hyperbolic geometry etc.

We can consider symmetric spaces from different points of view. In this paper we consider their algebraic features by considering Lie groups and their Lie algebras as algebraic approach to symmetric spaces. In fact a symmetric space can be considered as a Lie group G with a certain involution σ, or a homogeneous space $G/_H$ where G is a Lie group and H its isotropy subgroup. In the above sections we have discussed the important features and properties of Lie groups and their Lie algebras which help in disclosing some algebraic features of symmetric spaces. Also in this paper we cannot discuss all features such as types of symmetric spaces and their classification, but we gave introductory notions which help in future work in this field.

Involutive automorphism 9.1

Let $\mathbf{g}$ be a Lie algebra , the linear automorphism σ: $g \rightarrow \boldsymbol{g}$ is called an involutive automorphism if it satisfies $\sigma^2 = I$ (the identity) but $\sigma \neq I$, that is σ has eigenvalues ±1 and it splits the algebra $\boldsymbol{g}$ into orthogonal subspaces corresponding to these eigenvalues.

Symmetric subalgebra 9.2

If $\boldsymbol{g}$ is a compact simple Lie algebra **,** σ is an involutive automorphism of $\mathbf{g}$ and $\mathbf{g} = \mathfrak{h} \oplus \mathbf{P}$ satisfying $\sigma(X) = X$ for $X \in \mathfrak{h}$, $\sigma(X) = -X$ for $X \in P$ $\mathfrak{h}$ is a sub algebra , but $\boldsymbol{P}$ is not , and the following relations hold:

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} \quad , \quad [\mathfrak{h}, \mathbf{P}] \subset \mathbf{P}\, , \ [\mathbf{P}, \mathbf{P}] \subset \mathfrak{h} \qquad (9,2)$$

A subalgebra $\mathfrak{h}$ satisfying (9,2) is called symmetric subalgebra.

Cartan decomposition & symmetric spaces 9.3

Using what is known as Weyl unitary trick, that is by multiplying the elements in $\mathbf{P}$ by i we get a new noncompact algebra $\mathbf{g}^* = \mathfrak{h} \oplus i\mathbf{P}$, this is called a Cartan decomposition and $\mathfrak{h}$ is a maximal compact subalgebra of g^*. The Lie groups corresponding to the Lie algebras **g** & **g*** are G and H the isotropy subgroup of the Lie group G. Generally the coset space G/H is the set of subsets of G of the form gH, for $g \in G$, G acts on this coset space , that is the symmetric space.

Theorem 9.4 [9]

Any symmetric space S determines a Cartan decomposition on the Lie algebra of Killing fields. Vice versa, to any Lie algebra **g** with Cartan decomposition **g** = **𝔥** ⊕**P** there exists a unique simply connected symmetric space $S = G/H$ where G is the simply connected Lie group with Lie algebra g and H the connected subgroup with Lie algebra **𝔥**.

Example 9.5

Let $G = S \cup (n, C)$ be the group of unitary complex matrices with determinant +1. The algebra g= SU (n, ℂ) of this Lie group consists of complex antihermitian matrices of zero trace. $X \in g$ Can be written as $X = A + iB$ where A is real skew – symmetric and traceless and B is real, symmetric and traceless. Therefore **g = 𝔥 ⊕P** where **𝔥** is the compact connected subalgebra $SO(n, R)$ consisting of real, skew – symmetric and traceless matrices and **P** is the subspace of matrices of the form iB , where B is real, symmetric and traceless. **P** is not a sub algebra. g^* = **𝔥 ⊕iP** where **iP** is the subspace of real, symmetric and traceless matrices B. The Lie algebra $g^* = SL(n, R)$ is the set of $\boldsymbol{n} \times \boldsymbol{n}$ real matrices of zero trace and generates the linear group of transformations represented by real $\boldsymbol{n} \times \boldsymbol{n}$ matrices of unit determinant.

The involutive automorphism that splits the algebra **g** is defined by the complex conjugation σ= K, and for g^* the involutive automorphism is defined by

$\bar{\sigma} = (g^t)^{-1}$ for $\in g^*$. The decomposition g^*= **𝔥 ⊕iP** is the usual decomposition of a $SL(n, R)$ matrix in symmetric and skew – symmetric parts Now $G/_H$ = SU (n, ℂ) / $SO(n, R)$ is a symmetric space of compact type and the related symmetric space of non – compact type is

$$G^*/H = SL(n, R) \; / SO(n, R).$$

In this manner we can speak about different types of symmetric spaces especially for groups of matrices which has many applications.

10 More Features

Here we gave some notions of algebraic and geometric features of symmetric spaces. In fact a symmetric space is a Riemannian manifold in which the geodesic symmetry at each point is an isometry in a normal neighborhood of the point (Local property). Symmetric spaces are locally symmetric where the geodesic symmetries are global isometries.

Definition 10.1 (The rank)

The rank of a symmetric space M is the dimension of the largest abelian subalgebra of P , where g = **𝔥**⊕.

Theorem 10.2 [9]

A complete, locally symmetric , simply connected Riemannian manifold is a symmetric space.

Examples 10.3

The Euclidean n- Space E^n, The n- sphere S^n and the hyperbolic space H^n are standard examples of symmetric spaces, also these examples can be used for introducing more symmetric spaces and their properties.

Real forms in symmetric spaces 10.4

Real forms can be classified according to all the involutive automorphisms of the Lie algebra, satisfying $\sigma^2 = I$. We have two distinctive real forms which are the normal real form and the compact real form.

The normal real form of the algebra $\boldsymbol{g_c}$ which is also the least compact real form, consists of the subspaces containing real coefficients c^i & c^α. It has a metric with respect the bases $\{H_i, E_{\pm\alpha}\}$.

The compact real form of $\boldsymbol{g_c}$ is obtained by the Weyl unitary trick:

$$= \left\{\frac{(E_\alpha - E_{-\alpha})}{\sqrt{2}}\right\} . P = \left\{ iH_i , \frac{i(E_\alpha - E_{-\alpha})}{\sqrt{2}}\right\} \quad (10,1)$$

All real forms of any complex Lie algebra can be classified with characters lying between the character of the normal real form and the compact real form. This can be done just by enumerating all the involutive automorphisms of its compact real form **.** If $\mathbf{g}$ is the compact real form of a complex semisimple Lie algebrag_c g^* runs through all its associated noncompact real forms g^* ,' $*$, ... with corresponding maximal compact subgroups **𝔥, 𝔥'** and complementary subspaces $iP, i\,p', \ldots$ as σ runs through all the involutive automorphisms of $\mathbf{g}$. Also a complex algebra and all its real forms (the compact and all non-compact ones) correspond to the same root lattice and Dynkin diagram.

$$G^*/H \; = SL(n, R) \quad /SO(n, R)$$

Example 10.5

The normal real form of the complex algebra $\boldsymbol{g_c}$ $= SL(n, C)$ is the **non-compact algebra** $g^* = SL(n, R)$. This algebra can be decomposed as $\mathfrak{h} \oplus \boldsymbol{iP}$ where $\mathfrak{h}$ is the algebra consisting of real , skew – symmetric and traceless $n \times n$ matrices and $\boldsymbol{iP}$ is the algebra consisting of real , symmetric and traceless $n \times n$ matrices. Using the Weyl unitary trick, this algebra form the compact real form of $\boldsymbol{g_c}$, $Su(n, C)$ **=** $\boldsymbol{g}$ **=** $\mathfrak{h} \oplus \boldsymbol{iP}$**.**

Applying some involutive automorphisms to the elements of the compact real form $\boldsymbol{g}$, we can construct all the various non-compact real forms **g*, g'*,**

11 Main Results

1. The elements of a Lie group can act as transformations on the elements of the symmetric space.
2. If M is a symmetric space, its group of isometries G has a Lie group structure and we can obtain all information of M from G. If the point $p \in M$, H the isotropy subgroup at p and $\mathbf{g}$ is the Lie algebra of G , then the Lie algebra $\mathfrak{h}$ of H is a subalgebra of $\mathbf{g}$ having a complementary subspace $\boldsymbol{P}$ such that $\mathbf{g} = \mathfrak{h} \oplus \boldsymbol{P}$, $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, $[\mathfrak{h}, \boldsymbol{P}] \subset \boldsymbol{P}$ **and**

   $[\boldsymbol{P}, \boldsymbol{P}] \subset \boldsymbol{P}$ **,** and so the triple **(**$\mathbf{g}$ **,** $\mathfrak{h}$ **,** $\boldsymbol{P}$ **)** gives characterization of symmetric spaces.
3. Every Lie algebra corresponds to a given root system and each symmetric space corresponds to a restricted root system.
4. We can have several different spaces derived from the same Lie algebra.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Dyson F. Comm. Math. Phys. 1970;19:235.

[2] Olshanetsky MA, Perelomov AM. Phys. Rep. 1983;94:313.

[3] Hermann R. Lie Groups for Physicists (W. A. Benjamin Inc., New York); 1966.

[4] Ulrika Magnea. An introduction to symmetric spaces. Dept. of Math., University of Torino 10, 1 – 10125, Italy.

[5] Stephanie Kernik. A very brief overview of lie algebras. University of Minnesota, Morris; 2008.

[6] Humphreys, James E. Introduction to lie algebras and representation theory. Springer; 1994.

[7] Sattinger DH, Weaver OL. Lie groups & lie algebras with applications to physics, geometry and mechanics. Springer – Verlag, New York; 1986. ISBN; 3540962409.

[8] Boothby WM. An introduction to differentiable manifolds & Riemannian geometry. Academic Press, New York; 1975.

[9] Helgason S. Differential geometry, lie groups & symmetric spaces. Academic Press, New York; 1978. ISBN: 0-12-338460-5.

[10] Gilmore R. Lie groups, lie algebras and some of their applications. John Wiley & Sons, New York; 1974. ISBN: 0-471 – 30179 – 5.

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