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# On $\mathcal{G}^{\omega}$-Open Sets in Grill Topological Spaces

**Amin Saif[1*], Mohammed Al-Hawmi[2] and Basheer Al-Refaei[2]**

[1] *Department of Mathematics, Faculty of Sciences, Taiz University, Taiz, Yemen.*

[2] *Department of Mathematics, Faculty of Education, University of Saba Region, Mareb, Yemen.*

***Authors' contributions***

*This work was carried out in collaboration among all authors. Author AS designed the study and gave the relations and theorems of the study. Authors MAH and BAR wrote the first draft of the manuscript, managed the literature searches and investigated the relations and theorems of the study. All authors read and approved the final manuscript.*

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# Abstract

The propose of this paper is to introduce and investigate a weak form of $\omega$-open set in grill topological spaces. We introduce the notion of $\mathcal{G}^{\omega}$-open set as a form stronger than $\beta\omega$-open set and weaker than $\omega$-open set and $\mathcal{G}\beta$-open set. By using this form, we study the generalization property, the interior operator, closure operator and $\theta$-cluster operator.

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# 1 Introduction

In 1982 Hdeib [1], introduced the notion of $w$-open set as a weaker form of open set in topological spaces and by using this notion, [2] introduced the generalization property of $\omega$-open sets. In 1983 [3] introduced the notion of $\beta$-open set which is one of the famous weak forms of open set in

**Corresponding author: E-mail: alsanawyamin@yahoo.com;*

topological spaces. Under the notions of $\omega$-open sets and $\beta$-open sets, [4] introduced the notion of $\beta\omega$-open set as a weak form for $\omega$-open sets and $\beta$-open sets.

For the study of grill topological spaces, [5] introduced the concept of a grill on any nonempty sets. In 2007 [6] used the Kuratowski closure operator to define and introduce the concept of grill topological space. By using the notion of grill topological space, many mathematicians introduced and investigated weak and strong forms of open sets such us in 2011, [7] introduced the notion of $\mathcal{G}\beta$-open sets as a strong form of $\beta$-open sets.

In this paper, we introduce the notion of $\mathcal{G}^{\omega}$-open set as a form stronger than $\beta\omega$-open set and weaker than $\omega$-open set and $\mathcal{G}\beta$-open set. This paper is organized as follows. In Section 3, we introduce the concept of $\mathcal{G}^{\omega}$-open sets and we give its relationship with the other known sets. In Section 4, we study the interior operator, closure operator and $\theta$-cluster operator via the class of $\mathcal{G}^{\omega}$-open sets in grill topological spaces. In Section 5, we study and investigate the generalization property of $\mathcal{G}^{\omega}$-open sets.

# 2 Preliminaries

By $Cl(A)$ and $Int(A)$ we mean the closure set and the interior set of $A$ in topological space $(X, \tau)$, respectively.

**Theorem 2.1.** [8] For a topological space $(X, \tau)$ and $A, B \subseteq X$, if $B$ is an open set in $X$ then $Cl(A) \cap B \subseteq Cl(A \cap B)$.

**Theorem 2.2.** [8] For a topological space $(X, \tau)$,

1. $Cl(X - A) = X - Int(A)$ for all $A \subseteq X$.
2. $Int(X - A) = X - Cl(A)$ for all $A \subseteq X$.

**Definition 2.3.** [8] For a topological space $(X, \tau)$ and $E \subseteq X$, the *relativization topology* of $\tau$ to $E$ is denoted by $\tau|_E$ and defined by

$$\tau|_E = \{G \cap E : G \text{ is an open set in } X\}.$$

We say the pair $(E, \tau|_E)$ ia a subspace of $(X, \tau)$.

Let $(E, \tau|_E)$ be a subspace of a topological space $(X, \tau)$. For a subset $A$ of $E$, the $\tau|_E$-closure operator of $A$ is a set defined as the intersection of all closed subsets of $E$ containing $A$ and denoted by $Cl|_E(A)$. The $\tau|_E$-interior operator of $A$ is a set defined as the union of all open subsets of $E$ contained in $A$ and denoted by $Int|_E(A)$.

**Theorem 2.4.** [8] Let $(E, \tau|_E)$ be a subspace of a topological space $(X, \tau)$. For a subset $A$ of $E$:

1. $A$ is a closed in $E$ if and only if $A = F \cap E$ for some closed set $F$ in $X$.
2. $Cl|_E(A) = Cl(A) \cap E$.
3. $Int(A) \subseteq Int|_E(A)$.

**Definition 2.5.** [9] A topological space $(X, \tau)$ is called $T_{1/2}$*-space* if every $g$-closed set is a closed set.

**Theorem 2.6.** [10] A topological space $(X, \tau)$ is $T_{1/2}$*-space* if and only if every singleton set is open or closed set.

**Definition 2.7.** [11] Let $(X, \tau)$ be a topological space and $A \subseteq X$. A point $x \in X$ is called *$\theta$-cluster point* of $A$ if $Cl(U) \cap A \neq \emptyset$ for every open set $U$ in $X$ containing $x$.

The set of all $\theta$-cluster points of $A$ is called the *$\theta$-cluster set* of $A$ and denoted by $Cl^{\theta}(A)$. A subset $A$ of topological space is called *$\theta$-closed set* in $X$, [10], if $Cl^{\theta}(A) = A$. The complement of $\theta$-closed set in $X$ is called *$\theta$-open set* in $X$.

**Theorem 2.8.** [11] Every $\theta$-closed set is closed set.

**Definition 2.9.** [1] A subset $A$ of a space $X$ is called *$\omega$-open set* if for each $x \in A$, there is an open set $U_x$ containing $x$ such that $U_x - A$ is a countable set. The complement of $\omega$-open set is called *$\omega$-closed set.*

**Theorem 2.10.** [1] Every open set is $\omega$-open set.

**Definition 2.11.** [[3]] A subset $A$ of a space $X$ is called a *$\beta$-open* set if $A \subseteq Cl(Int(Cl(A)))$. The complement of $\beta$-open set is called $\beta$-closed set.

It is clear that every open set is $\beta$-open set.

**Definition 2.12.** *A function $f : (X, \tau) \to (Y, \rho)$ is a* $\beta$-continuous function *if $f^{-1}(U)$ is $\beta$-open set in $X$ for every open set $U$ in $Y$.*

Recall [[3]] that every continuous function is $\beta$-continuous function.

**Definition 2.13.** [4] A subset $A$ of a topological space $(X, \tau)$ is called $\beta\omega$-open set if $A \subseteq Cl(Int_{\omega}(Cl(A)))$. The complement of $\beta\omega$-open set is called $\beta\omega$-closed set.

**Theorem 2.14.** [4] Every $\omega$-open set is $\beta\omega$-open set and every $\beta$-open set is $\beta\omega$-open set.

**Definition 2.15.** [9] A subset $A$ of a topological space $(X, \tau)$ is called a *generalized closed* (simply $g$-closed) *set* if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is an open subset of $(X, \tau)$. The complement of $g$-closed set is called a *generalized open* (simply $g$-open) *set.*

**Theorem 2.16.** [9] Every closed set is $g$-closed set.

A collection $\mathcal{G}$ of subsets of a topological spaces $(X, \tau)$ is said to be a *grill* [5] on $X$ if $\mathcal{G}$ satisfies the following conditions:

1. $\emptyset \notin \mathcal{G}$;
2. $A \in \mathcal{G}$ and $A \subseteq B$ implies that $B \in \mathcal{G}$;
3. $A, B \subseteq X$ and $A \cup B \in \mathcal{G}$ implies that $A \in \mathcal{G}$ or $B \in \mathcal{G}$.

For a grill $\mathcal{G}$ on a topological space $X$, an operator from the power set $P(X)$ of $X$ to $P(X)$ was defined in [6] in the following manner : For any $A \in P(X)$,

$$\Phi(A) = \{x \in X : U \cap A \in \mathcal{G}, \text{ for each open neighborhood } U \text{ of } x\}.$$

Then the operator $\Psi : P(X) \to P(X)$, given by $\Psi(A) = A \cup \Phi(A)$, for $A \in P(X)$, was also shown in [6] to be a Kuratowski closure operator, defining a unique topology $\tau_{\mathcal{G}}$ on $X$ such that $\tau \subseteq \tau_{\mathcal{G}}$. This topology defined by

$$\tau_{\mathcal{G}} = \{U \subseteq X : \Psi(X - U) = X - U\},$$

where $\tau \subseteq \tau_{\mathcal{G}}$ and for any $A \subseteq X$, $\Psi(A) = {}_{\mathcal{G}}Cl(A)$ such that ${}_{\mathcal{G}}Cl(A)$ denotes the set of all closure points of $A$ in topological space $(X, \tau_{\mathcal{G}})$. The set of all interior points of $A$ in topological space $(X, \tau_{\mathcal{G}})$ denoted by ${}_{\mathcal{G}}Int(A)$.

If $(X, \tau)$ is a topological space and $\mathcal{G}$ is a grill on $X$ then the triple $(X, \tau, \mathcal{G})$ will be called a *grill topological space.*

**Theorem 2.17.** [6] Let $(X, \tau, \mathcal{G})$ be a grill topological space. Then for $A, B \subseteq X$, the following properties hold:

1. $A \subseteq B$ implies that $\Phi(A) \subseteq \Phi(B)$.
2. $\Phi(A \cup B) = \Phi(A) \cup \Phi(B)$.
3. $\Phi(\Phi(A)) \subseteq \Phi(A) = Cl(\Phi(A)) \subseteq Cl(A)$.
4. If $U \in \tau$ then $U \cap \Phi(A) \subseteq \Phi(U \cap A)$.

**Theorem 2.18.** [7] If $A$ is a subset of a grill topological space $(X, \tau, \mathcal{G})$ and $U$ is an open set in $(X, \tau)$ then $U \cap \Psi(A) \subseteq \Psi(U \cap A)$.

**Definition 2.19.** [7] A subset $A$ of a grill topological space $(X, \tau, \mathcal{G})$ is called *$\mathcal{G}\beta$-open set* if $A \subseteq Cl(Int(\Psi(A)))$. The complement of $\mathcal{G}\beta$-open set is called *$\mathcal{G}\beta$-closed set.*

Recall [7] that every open set in $(X, \tau)$ is $\mathcal{G}\beta$-open set in a grill topological space $(X, \tau, \mathcal{G})$ and every $\mathcal{G}\beta$-open set in $(X, \tau, \mathcal{G})$ is $\beta$-open set in $(X, \tau)$.

# 3 $\mathcal{G}^{\omega}$-Open Sets

For a topological space $(X, \tau)$ and $A \subseteq X$, the *$\omega$-closure operator* of $A$ is a set defined as the intersection of all $\omega$-closed subsets of $X$ containing $A$ and denoted by $Cl_{\omega}(A)$. The *$\omega$-interior operator* of $A$ is a set defined as the union of all $\omega$-open subsets of $X$ contained in $A$ and denoted by $Int_{\omega}(A)$.

**Definition 3.1.** A subset $G$ of grill topological space $(X, \tau, \mathcal{G})$ is called *$\mathcal{G}^{\omega}$-open set* if $G \subseteq Cl(Int_{\omega}(\Psi(G)))$. The complement of $\mathcal{G}^{\omega}$-open set is called *$\mathcal{G}^{\omega}$-closed set.*

In any grill topological space $(X, \tau, \mathcal{G})$ with a countable set $X$, it is clear that all subsets of $X$ are both $\mathcal{G}^{\omega}$-open sets and $\mathcal{G}^{\omega}$-closed sets. So any $\omega$-open set is $\mathcal{G}^{\omega}$-open set but the converse no need to be true. For example, let $(\mathbb{R}, \tau, \mathcal{G})$ be a grill topological space on the set of real numbers $\mathbb{R}$ with $\tau = \{\emptyset, \mathbb{R}, \mathbb{R} - \{1\}\}$ and $\mathcal{G} = P(\mathbb{R}) - \{\emptyset\}$. The set $\{2\}$ is $\mathcal{G}^{\omega}$-open set but not $\omega$-open set.

**Theorem 3.2.** Every $\mathcal{G}^{\omega}$-open set in a grill topological space $(X, \tau, \mathcal{G})$ is $\beta\omega$-open set in a space $(X, \tau)$.

*Proof.* Let $G$ be $\mathcal{G}^{\omega}$-open subset of a grill topological space $(X, \tau, \mathcal{G})$. Then

$$G \subseteq Cl(Int_{\omega}(\Psi(G))) \subseteq Cl(Int_{\omega}(Cl(G))).$$

That is, $G$ is $\beta\omega$-open set in a space $(X, \tau)$. □

The converse of above theorem no need to be true.

**Example 3.3.** Let $(\mathbb{R}, \tau, \mathcal{G})$ be a grill topological space on the set of real numbers $\mathbb{R}$ with $\tau = \{\emptyset, \mathbb{R}, \mathbb{R} - \{1\}\}$ and $\mathcal{G} = \{\mathbb{R}\}$. The set $\{2\}$ is $\beta\omega$-open set but not $\mathcal{G}^{\omega}$-open set.

**Theorem 3.4.** Every $\mathcal{G}\beta$-open set is $\mathcal{G}^{\omega}$-open set.

*Proof.* Similar for the proof of Theorem(3.2). □

The converse of above theorem no need to be true.

**Example 3.5.** Let $(X, \tau, \mathcal{G})$ be a grill topological space on the set $X = \{a, b, c\}$ with $\tau = \{\emptyset, X, \{a\}\}$ and $\mathcal{G} = P(X) - \{\emptyset\}$. The set $\{b, c\}$ is $\mathcal{G}^{\omega}$-open set but not $\mathcal{G}\beta$-open set.

**Theorem 3.6.** A subset $G$ of a grill topological space $(X, \tau, \mathcal{G})$ is $\mathcal{G}^{\omega}$-closed set if and only if $Int[Cl_{\omega}({}_{\mathcal{G}}Int(G))] \subseteq G$.

*Proof.* Let $G$ be any $\mathcal{G}^{\omega}$-closed set in grill topological space $(X, \tau, \mathcal{G})$. That is, $X - G$ is $\mathcal{G}^{\omega}$-open set in grill topological space $(X, \tau, \mathcal{G})$. Then we have

$$(X - G) \subseteq Cl[Int_{\omega}(\Psi(X - G))].$$

By using Theorems (2.2), this implies

$$\begin{aligned}(X - G) &\subseteq Cl[Int_{\omega}(\Psi(X - G))] = Cl[Int_{\omega}({}_{\mathcal{G}}Cl(X - G))] \\ &= Cl[Int_{\omega}(X - {}_{\mathcal{G}}Int(G))] = Cl[X - Cl_{\omega}({}_{\mathcal{G}}Int(G))] \\ &= X - Int[Cl_{\omega}({}_{\mathcal{G}}Int(G))].\end{aligned}$$

Hence $Int[Cl_{\omega}({}_{\mathcal{G}}Int(G))] \subseteq G$. The conversely is similar. □

**Theorem 3.7.** Let $(X, \tau, \mathcal{G})$ be a grill topological space. If $G_k$ is $\mathcal{G}^{\omega}$-open set for each $k \in I$ then $\cup_{k \in I} G_k$ is $\mathcal{G}^{\omega}$-open set, where $I$ is an index set.

*Proof.* Since $G_k$ is $\mathcal{G}^{\omega}$-open set for each $k \in I$ then $G_k \subseteq Cl[Int_{\omega}(\Psi(G_k))]$ for each $k \in I$. Then by Theorem (2.17),

$$\begin{aligned}\cup_{k \in I} G_k &\subseteq \cup_{k \in I} Cl[Int_{\omega}(\Psi(G_k))] \subseteq Cl[\cup_{k \in I} Int_{\omega}(\Psi(G_k))] \\ &\subseteq Cl[Int_{\omega}(\cup_{k \in I} \Psi(G_k))] \subseteq Cl[Int_{\omega}(\cup_{k \in I}(G_k \cup \Phi(G_k))] \\ &\subseteq Cl[Int_{\omega}((\cup_{k \in I} G_k) \cup (\cup_{k \in I} \Phi(G_k))] \\ &\subseteq Cl[Int_{\omega}(\cup_{k \in I} G_k \cup \Phi(\cup_{k \in I} G_k))] \\ &= Cl[Int_{\omega}(\Psi(\cup_{k \in I} G_k))].\end{aligned}$$

Hence $\cup_{k \in I} G_k$ is $\mathcal{G}^{\omega}$-open set. □

The intersection of two $\mathcal{G}^{\omega}$-open sets no need to be $\mathcal{G}^{\omega}$-open set.

**Example 3.8.** Let $(\mathbb{R}, \tau, \mathcal{G})$ be a grill topological space on the set of real numbers $\mathbb{R}$ with $\tau = \{\emptyset, \mathbb{R}, \mathbb{R} - \{1\}\}$ and $\mathcal{G} = P(\mathbb{R}) - \{\emptyset\}$. The sets $G = \{1, 2\}$ and $H = \{1, 3\}$ are $\mathcal{G}^{\omega}$-open sets but $G \cap H = \{1\}$ is not $\mathcal{G}^{\omega}$-open set.

**Theorem 3.9.** Let $(X, \tau, \mathcal{G})$ be a grill topological space. If $G$ is an open set in $(X, \tau)$ and $H$ is $\mathcal{G}^{\omega}$-open set then $G \cap H$ is $\mathcal{G}^{\omega}$-open set.

*Proof.* Since $H$ is $\mathcal{G}^{\omega}$-open set then $H \subseteq Cl[Int_{\omega}(\Psi(H))]$. Then by Theorems (2.17) and (2.1),

$$\begin{aligned}G \cap H &\subseteq G \cap Cl[Int_{\omega}(\Psi(H))] \subseteq Cl[G \cap Int_{\omega}(\Psi(H))] \\ &= Cl[Int_{\omega}(G) \cap Int_{\omega}(\Psi(H))] = Cl[Int_{\omega}(G \cap \Psi(H))] \\ &\subseteq Cl[Int_{\omega}(\Psi(G \cap H))].\end{aligned}$$

Hence $G \cap H$ is $\mathcal{G}^{\omega}$-open set. □

We mean by *bitopological space* is a triple $(X, \tau, \rho)$ consists two topologies $\tau$ and $\rho$ on a set $X$. A subset $G \subseteq X$ is said to be *$\omega(\tau, \rho)$-open set* in a bitopological space $(X, \tau, \rho)$ if $G \subseteq {}_{\tau}Cl[{}_{\tau}Int_{\omega}({}_{\rho}Cl(G))]$. The complement of $\omega(\tau, \rho)$-open set is said to be *$\omega(\tau, \rho)$-closed set.*

**Theorem 3.10.** A subset $G \subseteq X$ is $\mathcal{G}^{\omega}$-open set in grill topological space $(X, \tau, \mathcal{G})$ if and only if it is $\omega(\tau, \tau_{\mathcal{G}})$-open set in bitopological space $(X, \tau, \tau_{\mathcal{G}})$.

*Proof.* It is clear from the definitions and $\Psi(G) = {}_{\mathcal{G}}Cl(G) = {}_{\tau_{\mathcal{G}}}Cl(G)$. $\square$

**Theorem 3.11.** A subset $G$ of a bitopological space $(X, \tau, \rho)$ is $\omega(\tau, \rho)$-closed set if and only if ${}_{\tau}Int[{}_{\tau}Cl_{\omega}({}_{\rho}Int(G))] \subseteq G$.

*Proof.* Let $G$ be any $\omega(\tau, \rho)$-closed set in bitopological space $(X, \tau, \rho)$. That is, $X - G$ is $\omega(\tau, \rho)$-open set in bitopological space $(X, \tau, \rho)$. Hence

$$(X - G) \subseteq {}_{\tau}Cl[{}_{\tau}Int_{\omega}({}_{\rho}Cl(X - G))].$$

By using Theorems (2.2), we get that

$$\begin{aligned}(X - G) &\subseteq {}_{\tau}Cl[{}_{\tau}Int_{\omega}({}_{\rho}Cl(X - G))] = {}_{\tau}Cl[{}_{\tau}Int_{\omega}(X - {}_{\rho}Int(G))] \\ &= {}_{\tau}Cl[X - {}_{\tau}Cl_{\omega}({}_{\rho}Int(G))] = X - {}_{\tau}Int[{}_{\tau}Cl_{\omega}({}_{\rho}Int(G))].\end{aligned}$$

Hence ${}_{\tau}Int[{}_{\rho}Cl({}_{\rho}Int(G))] \subseteq G$. The conversely is similar. $\square$

**Theorem 3.12.** Let $E$ be an open subset of a grill topological space $(X, \tau, \mathcal{G})$. If $G$ is $\mathcal{G}^{\omega}$-open set in $(X, \tau, \mathcal{G})$ then $G \cap E$ is $\omega(\tau|_E, \tau_{\mathcal{G}}|_E)$-open set in bitopological space $(E, \tau|_E, \tau_{\mathcal{G}}|_E)$.

*Proof.* Since $G$ is $\mathcal{G}^{\omega}$-open set in $(X, \tau, \mathcal{G})$ then $G \subseteq Cl[Int_{\omega}(\Psi(G))]$. Then by Theorems (2.17), (2.4) and (2.1),

$$\begin{aligned}G \cap E &\subseteq Cl[Int_{\omega}(\Psi(G))] \cap E = Cl[Int_{\omega}(\Psi(G))] \cap E \cap E \\ &\subseteq Cl[Int_{\omega}(\Psi(G)) \cap E] \cap E = Cl|_E[Int_{\omega}(\Psi(G)) \cap E] \\ &= Cl|_E[Int_{\omega}(\Psi(G)) \cap Int_{\omega}(E)] = Cl|_E[Int_{\omega}(\Psi(G) \cap E)] \\ &= Cl|_E[Int_{\omega}(\Psi(G) \cap E \cap E)] \subseteq Cl|_E[Int_{\omega}(\Psi(G \cap E) \cap E)] \\ &\subseteq Cl|_E[Int_{\omega}|_E(\Psi(G \cap E) \cap E)] = Cl|_E[Int_{\omega}|_E({}_{\mathcal{G}}Cl(G \cap E) \cap E)] \\ &= Cl|_E[Int_{\omega}|_E({}_{\mathcal{G}}Cl|_E(G \cap E))].\end{aligned}$$

Hence $G \cap E$ is $\omega(\tau|_E, \tau_{\mathcal{G}}|_E)$-open set in $(E, \tau|_E, \tau_{\mathcal{G}}|_E)$. $\square$

**Corollary 3.13.** Let $E$ be an open subset of a grill topological space $(X, \tau, \mathcal{G})$. If $G$ is $\mathcal{G}^{\omega}$-closed set in $(X, \tau, \mathcal{G})$ then $G \cap E$ is $\omega(\tau|_E, \tau_{\mathcal{G}}|_E)$-closed set in in bitopological space $(E, \tau|_E, \tau_{\mathcal{G}}|_E)$.

*Proof.* Let $G$ be $\mathcal{G}^{\omega}$-closed set in $(X, \tau, \mathcal{G})$. Then $X - G$ is $\mathcal{G}^{\omega}$-open set in $(X, \tau, \mathcal{G})$. By the above theorem, $E - G = (X - G) \cap E$ is $\omega(\tau|_E, \tau_{\mathcal{G}}|_E)$-open set in $(E, \tau|_E, \tau_{\mathcal{G}}|_E)$. Hence

$$E - (E - G) = E - (E \cap (X - G)) = E \cap [(X - E) \cup G] = G \cap E$$

is $\omega(\tau|_E, \tau_{\mathcal{G}}|_E)$-closed set in $(E, \tau|_E, \tau_{\mathcal{G}}|_E)$. $\square$

**Theorem 3.14.** Let $E$ be an open subset of a grill topological space $(X, \tau, \mathcal{G})$. If $G$ is $\omega(\tau|_E, \tau_{\mathcal{G}}|_E)$-open set in bitopological space $(E, \tau|_E, \tau_{\mathcal{G}}|_E)$ then $G$ is $\mathcal{G}^{\omega}$-open set in $(X, \tau, \mathcal{G})$.

*Proof.* Since $G$ is $\omega(\tau|_E, \tau_{\mathcal{G}}|_E)$-open set in $(E, \tau|_E, \tau_{\mathcal{G}}|_E)$ then

$$G \subseteq Cl|_E[Int_{\omega}|_E({}_{\mathcal{G}}Cl|_E(G))].$$

Then by Theorems (2.17), (2.4) and (2.1),

$$\begin{aligned}G &\subseteq Cl|_E[Int_{\omega}|_E({}_{\mathcal{G}}Cl|_E(G))] = Cl[Int_{\omega}|_E({}_{\mathcal{G}}Cl|_E(G))] \cap E \\ &\subseteq Cl[Int_{\omega}|_E({}_{\mathcal{G}}Cl|_E(G)) \cap E] = Cl[Int_{\omega}|_E({}_{\mathcal{G}}Cl|_E(G))] \\ &= Cl[Int_{\omega}({}_{\mathcal{G}}Cl|_E(G))] = Cl[Int_{\omega}({}_{\mathcal{G}}Cl|(G) \cap E)] \\ &\subseteq Cl[Int_{\omega}({}_{\mathcal{G}}Cl(G \cap E))] = Cl[Int_{\omega}({}_{\mathcal{G}}Cl(G))] \\ &= Cl[Int_{\omega}(\Psi(G))].\end{aligned}$$

Hence $G$ is $\mathcal{G}^{\omega}$-open set $G$ in $(X, \tau, \mathcal{G})$. $\square$

**Corollary 3.15.** Let $E$ be an open subset of a grill topological space $(X, \tau, \mathcal{G})$. If $G$ is $\omega(\tau|_E, \tau_{\mathcal{G}}|_E)$-closed set in bitopological space $(E, \tau|_E, \tau_{\mathcal{G}}|_E)$ then $G$ is $\mathcal{G}^\omega$-closed set in $(X, \tau, \mathcal{G})$.

# 4 $\mathcal{G}^\omega$-Operators

**Definition 4.1.** Let $(X, \tau, \mathcal{G})$ be a grill topological space and $G \subseteq X$.

1. The $\mathcal{G}^\omega$*-closure operator* of $G$ is denoted by ${}_{\mathcal{G}^\omega}Cl(G)$ and defined by
$$ {}_{\mathcal{G}^\omega}Cl(G) = \cap\{H \subseteq X : G \subseteq H \text{ and } H \in \mathcal{G}^\omega_C(X, \tau)\}. $$
That is, ${}_{\mathcal{G}^\omega}Cl(G)$ is the intersection of all $\mathcal{G}^\omega$-closed sets containing $G$.
2. The $\mathcal{G}^\omega$*-interior operator* of $G$ is denoted by ${}_{\mathcal{G}^\omega}Int(G)$ and defined by
$$ {}_{\mathcal{G}^\omega}Int(G) = \cup\{H \subseteq X : H \subseteq G \text{ and } H \in \mathcal{G}^\omega_O(X, \tau)\}. $$
That is, ${}_{\mathcal{G}^\omega}Int(G)$ is the union of all $\mathcal{G}^\omega$-open sets contained in $G$.
3. The $\theta{-}\mathcal{G}^\omega$*-cluster operator* of $G$ is defined by the set of all $\theta{-}\mathcal{G}^\omega$-cluster points of $G$ and denoted by ${}_{\mathcal{G}^\omega}Cl^\theta(G)$. A point $x \in X$ is called $\theta{-}\mathcal{G}^\omega$*-cluster point* of $G$ if ${}_{\mathcal{G}^\omega}Cl(U) \cap G \neq \emptyset$ for every $\mathcal{G}^\omega$-open set $U$ in $(X, \tau, \mathcal{G})$ containing $x$.

**Theorem 4.2.** Let $(X, \tau, \mathcal{G})$ be a grill topological space and $G \subseteq X$. Then ${}_{\mathcal{G}^\omega}Int(G) = G$ if and only if $G$ is a $\mathcal{G}^\omega$-open set.

*Proof.* Let ${}_{\mathcal{G}^\omega}Int(G) = G$. Then from definition of ${}_{\mathcal{G}^\omega}Int(G)$ and Theorem (3.7), ${}_{\mathcal{G}^\omega}Int(G)$ is $\mathcal{G}^\omega$-open set and so $G$ is $\mathcal{G}^\omega$-open set.

Conversely, we have ${}_{\mathcal{G}^\omega}Int(G) \subseteq G$ by the definition. Since $G$ is a $\mathcal{G}^\omega$-open set, then it is clear from the definition of ${}_{\mathcal{G}^\omega}Int(G)$, $G \subseteq {}_{\mathcal{G}^\omega}Int(G)$. Hence $G = {}_{\mathcal{G}^\omega}Int(G)$. □

**Theorem 4.3.** Let $(X, \tau, \mathcal{G})$ be a grill topological space and $G \subseteq X$. Then ${}_{\mathcal{G}^\omega}Cl(G) = G$ if and only if $G$ is a $\mathcal{G}^\omega$-closed set.

*Proof.* Similar for proof of Theorem (4.2). □

**Theorem 4.4.** Let $(X, \tau, \mathcal{G})$ be a grill topological space and $G \subseteq X$. Then $x \in {}_{\mathcal{G}^\omega}Int(G)$ if and only if there is $\mathcal{G}^\omega$-open set $U$ such that $x \in U \subseteq G$.

*Proof.* Let $x \in {}_{\mathcal{G}^\omega}Int(G)$ and take $U = {}_{\mathcal{G}^\omega}Int(G)$. Then by Theorem (3.7) and definition of ${}_{\mathcal{G}^\omega}Int(G)$ we get that $U$ is a $\mathcal{G}^\omega$-open set and $x \in U \subseteq G$.

Conversely, Let there is $\mathcal{G}^\omega$-open set $U$ such that $x \in U \subseteq G$. Then by definition of ${}_{\mathcal{G}^\omega}Int(G)$, $x \in U \subseteq {}_{\mathcal{G}^\omega}Int(G)$. □

**Theorem 4.5.** Let $(X, \tau, \mathcal{G})$ be a grill topological space and $G \subseteq X$. Then $x \in {}_{\mathcal{G}^\omega}Cl(G)$ if and only if for all $\mathcal{G}^\omega$-open set $U$ containing $x$, $U \cap G \neq \emptyset$.

*Proof.* Let $x \in {}_{\mathcal{G}^\omega}Cl(G)$ and $U$ be $\mathcal{G}^\omega$-open set containing $x$. If $U \cap G = \emptyset$ then $G \subseteq X - U$. Since $X - U$ is a $\mathcal{G}^\omega$-closed set containing $G$, then ${}_{\mathcal{G}^\omega}Cl(G) \subseteq X - U$ and so $x \in {}_{\mathcal{G}^\omega}Cl(G) \subseteq X - U$. This is contradiction, because $x \in U$. Therefore $U \cap G \neq \emptyset$.

Conversely, Let $x \notin {}_{\mathcal{G}^\omega}Cl(G)$. Then $X - {}_{\mathcal{G}^\omega}Cl(G)$ is $\mathcal{G}^\omega$-open set containing $x$. Hence by hypothesis, $[X - {}_{\mathcal{G}^\omega}Cl(G)] \cap G \neq \emptyset$. This is contradiction, because $X - {}_{\mathcal{G}^\omega}Cl(G) \subseteq X - G$. □

**Theorem 4.6.** Let $(X, \tau, \mathcal{G})$ be a grill topological space and $G, H \subseteq X$. Then the following hold:

1. If $G \subseteq H$ then ${}_{\mathcal{G}^\omega}Int(G) \subseteq {}_{\mathcal{G}^\omega}Int(H)$.
2. ${}_{\mathcal{G}^\omega}Int(G) \cup {}_{\mathcal{G}^\omega}Int(H) \subseteq {}_{\mathcal{G}^\omega}Int(G \cup H)$.
3. ${}_{\mathcal{G}^\omega}Int(G \cap H) \subseteq {}_{\mathcal{G}^\omega}Int(G) \cap {}_{\mathcal{G}^\omega}Int(H)$.
4. $Int(G) \subseteq {}_{\mathcal{G}^\omega}Int(G)$ .

In the last theorem ${}_{\mathcal{G}^\omega}Int(G \cap H) \neq {}_{\mathcal{G}^\omega}Int(G) \cap {}_{\mathcal{G}^\omega}Int(H)$.

**Example 4.7.** In Example (3.8), the sets $G = \{1, 2\}$ and $H = \{1, 3\}$ are $\mathcal{G}^\omega$-open sets in $(X, \tau, \mathcal{G})$. Then ${}_{\mathcal{G}^\omega}Int(G) \cap {}_{\mathcal{G}^\omega}Int(H) = G \cup H = \{1\}$ and

$${}_{\mathcal{G}^\omega}Int(G \cap H) = {}_{\mathcal{G}^\omega}Int(\{1\}) = \emptyset.$$

**Theorem 4.8.** Let $(X, \tau, \mathcal{G})$ be a grill topological space and $G, H \subseteq X$. Then the following hold:

1. If $G \subseteq H$ then ${}_{\mathcal{G}^\omega}Cl(G) \subseteq {}_{\mathcal{G}^\omega}Cl(H)$.
2. ${}_{\mathcal{G}^\omega}Cl(G) \cup {}_{\mathcal{G}^\omega}Cl(H) \subseteq {}_{\mathcal{G}^\omega}Cl(G \cup H)$.
3. ${}_{\mathcal{G}^\omega}Cl(G \cap H) \subseteq {}_{\mathcal{G}^\omega}Cl(G) \cap {}_{\mathcal{G}^\omega}Cl(H)$.
4. ${}_{\mathcal{G}^\omega}Cl(G) \subseteq Cl(G)$.

In the last theorem ${}_{\mathcal{G}^\omega}Cl(G \cup H) \neq {}_{\mathcal{G}^\omega}Cl(G) \cup {}_{\mathcal{G}^\omega}Cl(H)$.

**Example 4.9.** In Example (3.8), the sets $G = \mathbb{R} - \{1, 2\}$ and $H = \mathbb{R} - \{1, 3\}$ are $\mathcal{G}^\omega$-closed sets in $(X, \tau, \mathcal{G})$. Then ${}_{\mathcal{G}^\omega}Cl(G) \cup {}_{\mathcal{G}^\omega}Cl(H) = G \cup H = \mathbb{R} - \{1\}$ and

$${}_{\mathcal{G}^\omega}Cl(G \cup H) = {}_{\mathcal{G}^\omega}Cl(\mathbb{R} - \{1\}) = \mathbb{R}.$$

**Theorem 4.10.** Let $(X, \tau, \mathcal{G})$ be a grill topological space and $G \subseteq X$. Then the following hold:

1. ${}_{\mathcal{G}^\omega}Int(X - G) = X - {}_{\mathcal{G}^\omega}Cl(G)$.
2. ${}_{\mathcal{G}^\omega}Cl(X - G) = X - {}_{\mathcal{G}^\omega}Int(G)$.

*Proof.* 1. Since $G \subseteq {}_{\mathcal{G}^\omega}Cl(G)$ then $X - {}_{\mathcal{G}^\omega}Cl(G) \subseteq X - G$. Since $X - {}_{\mathcal{G}^\omega}Cl(G)$ is $\mathcal{G}^\omega$-open set in $(X, \tau, \mathcal{G})$ then

$$X - {}_{\mathcal{G}^\omega}Cl(G) = {}_{\mathcal{G}^\omega}Int[X - {}_{\mathcal{G}^\omega}Cl(G)] \subseteq {}_{\mathcal{G}^\omega}Int(X - G).$$

For the other side, let $x \in {}_{\mathcal{G}^\omega}Int(X - G)$. Then there is $\mathcal{G}^\omega$-open set $U$ such that $x \in U \subseteq X - G$. Then $X - U$ is $\mathcal{G}^\omega$-closed set containing $G$ and $x \notin X - U$. Hence $x \notin {}_{\mathcal{G}^\omega}Cl(G)$, that is, $x \in X - {}_{\mathcal{G}^\omega}Cl(G)$.
2. Since ${}_{\mathcal{G}^\omega}Int(G) \subseteq G$ then $X - G \subseteq X - {}_{\mathcal{G}^\omega}Int(G)$. Since $X - {}_{\mathcal{G}^\omega}Int(G)$ is $\mathcal{G}^\omega$-closed set in $(X, \tau, \mathcal{G})$ then

$${}_{\mathcal{G}^\omega}Cl(X - G) \subseteq {}_{\mathcal{G}^\omega}Cl[X - {}_{\mathcal{G}^\omega}Int(G)] = X - {}_{\mathcal{G}^\omega}Int(G).$$

For the other side, let $x \in {}_{\mathcal{G}^\omega}Int(X - G)$. Then there is $\mathcal{G}^\omega$-open set $U$ such that $x \in U \subseteq X - G$. Then $X - U$ is $\mathcal{G}^\omega$-closed set containing $G$ and $x \notin X - U$. Hence $x \notin {}_{\mathcal{G}^\omega}Cl(G)$, that is, $x \in X - {}_{\mathcal{G}^\omega}Cl(G)$. ☐

**Theorem 4.11.** Let $(X, \tau, \mathcal{G})$ be a grill topological space and $G \subseteq X$. Then the following hold:

1. If $H$ is an open set $X$ then ${}_{\mathcal{G}^\omega}Cl(G) \cap H \subseteq {}_{\mathcal{G}^\omega}Cl(G \cap H)$.
2. If $H$ is a closed set $X$ then ${}_{\mathcal{G}^\omega}Int(G \cup H) \subseteq {}_{\mathcal{G}^\omega}Int(G) \cup H$.

*Proof.* (1) Let $x \in {}_{\mathcal{G}^\omega}Cl(G) \cap H$. Then $x \in {}_{\mathcal{G}^\omega}Cl(G)$ and $x \in H$. Let $V$ be any $\mathcal{G}^\omega$-open set in $(X, \tau, \mathcal{G})$ containing $x$. By Theorem (3.9), $V \cap H$ is $\mathcal{G}^\omega$-open set containing $x$. Since $x \in {}_{\mathcal{G}^\omega}Cl(G)$ then by Theorem (4.5), $(V \cap H) \cap G \neq \emptyset$. This implies, $V \cap (H \cap G) \neq \emptyset$. Hence by Theorem (4.5), $x \in {}_{\mathcal{G}^\omega}Cl(G \cap H)$. That is, ${}_{\mathcal{G}^\omega}Cl(G) \cap H \subseteq {}_{\mathcal{G}^\omega}Cl(G \cap H)$. (2) Since $H$ is a closed set $X$ then by the part (1) and Theorem (4.10),

$$\begin{aligned} X - [{}_{\mathcal{G}^\omega}Int(G) \cup H] &= [X - {}_{\mathcal{G}^\omega}Int(G)] \cap [X - H] \\ &= [{}_{\mathcal{G}^\omega}Cl(X - G)] \cap [X - H] \\ &\subseteq {}_{\mathcal{G}^\omega}Cl[(X - G) \cap (X - H)] \\ &\subseteq {}_{\mathcal{G}^\omega}Cl(X - G) \cap {}_{\mathcal{G}^\omega}Cl(X - H) \\ &= {}_{\mathcal{G}^\omega}Cl(X - G) \cap (X - H) \\ &= (X - {}_{\mathcal{G}^\omega}Int(G)) \cap (X - H) \\ &= X - ({}_{\mathcal{G}^\omega}Int(G)) \cup H). \end{aligned}$$

Hence ${}_{\mathcal{G}^\omega}Int(G \cup H) \subseteq {}_{\mathcal{G}^\omega}Int(G) \cup H$. □

**Theorem 4.12.** Let $E$ be an open subset of a grill topological space $(X, \tau, \mathcal{G})$ and $A$ be a subset of $E$. Then ${}_{\mathcal{G}^\omega}Cl|_E(G) = {}_{\mathcal{G}^\omega}Cl(G) \cap E$.

*Proof.* Let $x \in {}_{\mathcal{G}^\omega}Cl|_E(G)$ and $H$ be $\mathcal{G}^\omega$-open set in $X$ containing $x$. By Theorem (3.12), $H \cap E$ is $\omega(\tau, \tau_{\mathcal{G}})$-open set in bitopological space $(E, \tau|_E, \tau_{\mathcal{G}}|_E)$ containing $x$ and since $x \in {}_{\mathcal{G}^\omega}Cl|_E(G)$, then $G \cap H = (G \cap E) \cap H \neq \emptyset$. Hence by Theorem (4.5), $x \in {}_{\mathcal{G}^\omega}Cl(G)$, and since $x \in E$, this implies $x \in {}_{\mathcal{G}^\omega}Cl(G) \cap E$. That is, ${}_{\mathcal{G}^\omega}Cl|_E(G) \subseteq {}_{\mathcal{G}^\omega}Cl(G) \cap E$.

On the other side, let $x \in {}_{\mathcal{G}^\omega}Cl(G) \cap E$ and $O$ be $\omega(\tau, \tau_{\mathcal{G}})$-open set in bitopological space $(E, \tau|_E, \tau_{\mathcal{G}}|_E)$ containing $x$. By Corollary (3.15), $O$ is $\mathcal{G}^\omega$-open set in $(X, \tau, \mathcal{G})$. Since $x \in {}_{\mathcal{G}^\omega}Cl(G)$, then $O \cap G \neq \emptyset$. That is, $x \in {}_{\mathcal{G}^\omega}Cl|_E(G)$. Hence ${}_{\mathcal{G}^\omega}Cl(G) \cap E \subseteq {}_{\mathcal{G}^\omega}Cl|_E(G)$. □

**Definition 4.13.** A subset $G$ of grill topological space $(X, \tau, \mathcal{G})$ is called $\theta{-}\mathcal{G}^\omega$*-closed set* in $(X, \tau, \mathcal{G})$ if ${}_{\mathcal{G}^\omega}Cl^\theta(G) = G$. The complement of $\theta{-}\mathcal{G}^\omega$-closed set in $(X, \tau, \mathcal{G})$ is called $\theta{-}\mathcal{G}^\omega$*-open set* in $(X, \tau, \mathcal{G})$.

**Theorem 4.14.** Every $\theta$-closed set in a space $(X, \tau)$ is $\theta{-}\mathcal{G}^\omega$-closed set in grill topological space $(X, \tau, \mathcal{G})$.

*Proof.* Let $G$ be a $\theta$-closed set in a space $(X, \tau)$, that is, $Cl^\theta(G) = G$. It is clear that $G \subseteq {}_{\mathcal{G}^\omega}Cl^\theta(G)$. We prove that ${}_{\mathcal{G}^\omega}Cl^\theta(G) \subseteq G$. Let $x \in {}_{\mathcal{G}^\omega}Cl^\theta(G)$. Then ${}_{\mathcal{G}^\omega}Cl(U) \cap G \neq \emptyset$ for every $\mathcal{G}^\omega$-open set $U$ in $(X, \tau, \mathcal{G})$ containing $x$. Since ${}_{\mathcal{G}^\omega}Cl(U) \subseteq Cl(U)$ then $Cl(U) \cap G \neq \emptyset$ for every $\mathcal{G}^\omega$-open set $U$ in $(X, \tau, \mathcal{G})$ containing $x$. Then $x \in Cl^\theta(G) = G$. Hence ${}_{\mathcal{G}^\omega}Cl^\theta(G) \subseteq G$. That is, $G$ is $\theta{-}\mathcal{G}^\omega$-closed set in grill topological space $(X, \tau, \mathcal{G})$. □

The converse of the last theorem need not be true.

**Example 4.15.** In a grill topological space $(X, \tau, \mathcal{G})$, where $X = \{1, 2, 3\}$,

$$\tau = \{\emptyset, X, \{1, 2\}\} \text{ and } \mathcal{G} = \{\{3\}, \{1, 3\}, \{2, 3\}, X\},$$

the set $\{1\}$ is $\theta{-}\mathcal{G}^\omega$-closed set in $(X, \tau, \mathcal{G})$ but it is not $\theta$-closed set in $(X, \tau)$.

**Theorem 4.16.** Every $\theta{-}\mathcal{G}^\omega$-closed set is $\mathcal{G}^\omega$-closed set.

*Proof.* Let $(X, \tau, \mathcal{G})$ be a grill topological space and $G$ be $\theta{-}\mathcal{G}^\omega$-closed set, that is, ${}_{\mathcal{G}^\omega}Cl^\theta(G) = G$. It is clear that $G \subseteq {}_{\mathcal{G}^\omega}Cl(G)$. We prove that ${}_{\mathcal{G}^\omega}Cl(G) \subseteq G$. Let $x \in {}_{\mathcal{G}^\omega}Cl(G)$. Then $U \cap G \neq \emptyset$ for every $\mathcal{G}^\omega$-open set $U$ in $(X, \tau, \mathcal{G})$ containing $x$. Since $U \subseteq {}_{\mathcal{G}^\omega}Cl(U)$ then ${}_{\mathcal{G}^\omega}Cl(U) \cap G \neq \emptyset$ for every $\mathcal{G}^\omega$-open set $U$ in $(X, \tau, \mathcal{G})$ containing $x$. Then $x \in {}_{\mathcal{G}^\omega}Cl^\theta(G) = G$. Hence ${}_{\mathcal{G}^\omega}Cl(G) \subseteq G$. That is, $G$ is $\mathcal{G}^\omega$-closed set in grill topological space $(X, \tau, \mathcal{G})$. □

**Theorem 4.17.** For $\mathcal{G}^\omega$-open set $G$ in grill topological space $(X, \tau, \mathcal{G})$, ${}_{\mathcal{G}^\omega}Cl^\theta(G) = {}_{\mathcal{G}^\omega}Cl(G)$.

*Proof.* Let $x \in {}_{\mathcal{G}^\omega}Cl(G)$. Then for every $\mathcal{G}^\omega$-open set $U$ in $(X, \tau, \mathcal{G})$ containing $x$, $U \cap G \neq \emptyset$. Since $U \subseteq {}_{\mathcal{G}^\omega}Cl(U)$ then ${}_{\mathcal{G}^\omega}Cl(U) \cap G \neq \emptyset$. Hence $x \in {}_{\mathcal{G}^\omega}Cl(U)$. That is, ${}_{\mathcal{G}^\omega}Cl(G) \subseteq {}_{\mathcal{G}^\omega}Cl^\theta(G)$. For the other side, let $x \in {}_{\mathcal{G}^\omega}Cl^\theta(G)$. Then for every $\mathcal{G}^\omega$-open set $U$ in $(X, \tau, \mathcal{G})$ containing $x$, ${}_{\mathcal{G}^\omega}Cl(U) \cap G \neq \emptyset$. Since $G$ is $\mathcal{G}^\omega$-open set $U$ in $(X, \tau, \mathcal{G})$ then by Theorem (4.11), ${}_{\mathcal{G}^\omega}Cl(U) \cap G = {}_{\mathcal{G}^\omega}Cl(U \cap G)$. Then ${}_{\mathcal{G}^\omega}Cl(U \cap G) \neq \emptyset$. Hence $U \cap G \neq \emptyset$. That is, $x \in {}_{\mathcal{G}^\omega}Cl(U)$. That is, $[{}_{\mathcal{G}^\omega}Cl^\theta(G) \subseteq {}_{\mathcal{G}^\omega}Cl(G)$. ☐

**Theorem 4.18.** A subset $U$ is $\theta{-}\mathcal{G}^\omega$-open set in grill topological space $(X, \tau, \mathcal{G})$ if and only if for each $x \in U$ there is $\mathcal{G}^\omega$-open set $V$ in $(X, \tau, \mathcal{G})$ containing $x$ such that ${}_{\mathcal{G}^\omega}Cl(V) \subseteq U$.

*Proof.* Suppose that $U$ is $\theta{-}\mathcal{G}^\omega$-open set in $(X, \tau, \mathcal{G})$ and $x \in U$. Then

$$x \notin X - U = {}_{\mathcal{G}^\omega}Cl^\theta(X - U).$$

Then there is $\mathcal{G}^\omega$-open set $V$ in $(X, \tau, \mathcal{G})$ containing $x$ such that ${}_{\mathcal{G}^\omega}Cl(V) \cap (X - U) = \emptyset$; That is, ${}_{\mathcal{G}^\omega}Cl(V) \subseteq U$.

Conversely, suppose that $U$ is not $\theta{-}\mathcal{G}^\omega$-open set. Then $X - U$ is not $\theta{-}\mathcal{G}^\omega$-closed set. That is, there is $x \in {}_{\mathcal{G}^\omega}Cl^\theta(X - U)$ and $x \notin X - U$. Since $x \in U$ then by the hypothesis, there is $\mathcal{G}^\omega$-open set $V$ in $(X, \tau, \mathcal{G})$ containing $x$ such that ${}_{\mathcal{G}^\omega}Cl(V) \cap U$. This implies, ${}_{\mathcal{G}^\omega}Cl(V) \cap (X - U) = \emptyset$ and this contradiction since $x \in {}_{\mathcal{G}^\omega}Cl^\theta(X - U)$.. Hence $U$ is $\theta{-}\mathcal{G}^\omega$-open set. ☐

# 5 Generalized $\mathcal{G}^\omega$-open Sets

In this section, we study the generalization property of $\mathcal{G}^\omega$-open sets by giving the weak form of $\beta\omega-$open set, called generalized $\beta\omega-$open set.

**Definition 5.1.** A subset $G$ of a grill topological space $(X, \tau, \mathcal{G})$ is called *generalized $\mathcal{G}^\omega$-closed set* (simply $\mathcal{G}^\omega_g$-closed) if ${}_{\mathcal{G}^\omega}Cl(G) \subseteq U$ whenever $G \subseteq U$ and $U$ is open subset of $(X, \tau, \mathcal{G})$. The complement of $\mathcal{G}^\omega_g$-closed set is called *generalized $\mathcal{G}^\omega$-open set* (simply $\mathcal{G}^\omega_g$-open).

**Theorem 5.2.** Every $\mathcal{G}^\omega$-open set is $\mathcal{G}^\omega_g$-open set.

*Proof.* Let $G$ be $\mathcal{G}^\omega$-open subset of a grill topological space $(X, \tau, \mathcal{G})$. Then $X - G$ is $\mathcal{G}^\omega$-closed set. Hence $X - G = {}_{\mathcal{G}^\omega}Cl(X - G) \subseteq U$ whenever $X - G \subseteq U$ and $U$ is open set. That is, $G$ is $\mathcal{G}^\omega_g$-open set. ☐

The converse of above theorem no need to be true.

**Example 5.3.** Let $(\mathbb{R}, \tau, \mathcal{G})$ be a grill topological space on the set of real numbers $\mathbb{R}$ with $\tau = \{\emptyset, \mathbb{R}\}$ and $\mathcal{G} = \{\mathbb{R}\}$. The set $\{1, 2\}$ is $\mathcal{G}^\omega_g$-open set and not $\mathcal{G}^\omega$-open set.

**Theorem 5.4.** Let $(X, \tau, \mathcal{G})$ be a grill topological space. Every every $g\omega$-open set in $(X, \tau)$ is $\mathcal{G}^\omega_g$-open set.

*Proof.* It is clear that every $\omega$-open set is $\mathcal{G}^\omega$-open set and ${}_{\mathcal{G}^\omega}Cl(G) \subseteq Cl_\omega(G)$. ☐

The converse of above theorem no need to be true.

**Example 5.5.** Let $(\mathbb{R}, \tau, \mathcal{G})$ be a grill topological space on the set of real numbers $\mathbb{R}$ with $\tau = \{\emptyset, \mathbb{R}, \mathbb{R} - \{1\}\}$ and $\mathcal{G} = \{\mathbb{R}\}$. Since $\mathbb{R} - \{1\}$ is an open set containing itself, then the set $\{1\}$ is $\mathcal{G}^\omega_g$-open set in $(\mathbb{R}, \tau, \mathcal{G})$ and not $g\omega$-open set in $(\mathbb{R}, \tau)$.

**Theorem 5.6.** Let $(X, \tau, \mathcal{G})$ be a grill topological space. If $(X, \tau)$ is $T_{1/2}$-space then every $\mathcal{G}_g^\omega$-closed set in $(X, \tau, \mathcal{G})$ is $\mathcal{G}^\omega$-closed set.

*Proof.* Let $G$ be $\mathcal{G}_g^\omega$-closed set in $(X, \tau, \mathcal{G})$. Suppose that $G$ is not $\mathcal{G}^\omega$-closed set. Then there is at least $x \in {}_{\mathcal{G}^\omega}Cl(G)$ such that $x \notin G$. Since $(X, \tau)$ is $T_{1/2}$-space then by Theorem (2.6), $\{x\}$ is an open or closed set in $X$. If $\{x\}$ is a closed set in $X$ then $X - \{x\}$ is an open. Since $x \notin G$ then $G \subseteq X - \{x\}$. Since $G$ is $\mathcal{G}_g^\omega$-closed set and $X - \{x\}$ is an open subset of $X$ containing $G$, then ${}_{\mathcal{G}^\omega}Cl(G) \subseteq X - \{x\}$. Hence $x \in X - {}_{\mathcal{G}^\omega}Cl(G)$ and this a contradiction, since $x \in {}_{\mathcal{G}^\omega}Cl(G)$. If $\{x\}$ is an open set then it is $\mathcal{G}^\omega$-open set. Since $x \in {}_{\mathcal{G}^\omega}Cl(G)$ then we have $\{x\} \cap G \neq \emptyset$. That is, $x \in G$ and this a contradiction. Hence $G$ is $\mathcal{G}^\omega$-closed set in $(X, \tau, \mathcal{G})$. □

**Theorem 5.7.** If $G$ is $\mathcal{G}_g^\omega$-closed set in a grill topological space $(X, \tau, \mathcal{G})$ and $H$ is a closed set in $(X, \tau)$ then $G \cap H$ is $\mathcal{G}_g^\omega$-closed set.

*Proof.* Let $U$ be an open subset of $(X, \tau)$ such that $G \cap H \subseteq U$. Since $H$ is a closed set in $(X, \tau)$ then $U \cup (X - H)$ is an open set in $(X, \tau)$. Since $G$ is $\mathcal{G}_g^\omega$-closed set in $X$ and $G \subseteq U \cup (X - H)$ then ${}_{\mathcal{G}^\omega}Cl(G) \subseteq U \cup (X - H)$. Hence

$$\begin{aligned} {}_{\mathcal{G}^\omega}Cl(G \cap H) &\subseteq {}_{\mathcal{G}^\omega}Cl(G) \cap {}_{\mathcal{G}^\omega}Cl(H) \subseteq {}_{\mathcal{G}^\omega}Cl(G) \cap Cl(H) \\ &= {}_{\mathcal{G}^\omega}Cl(G) \cap H \subseteq [U \cup (X - H)] \cap H \\ &\subseteq U \cap H \subseteq U. \end{aligned}$$

Thus, $G \cap H$ is $\mathcal{G}_g^\omega$-closed set. □

**Theorem 5.8.** For any $x \in X$ in a grill topological space $(X, \tau, \mathcal{G})$, either the set $\{x\}$ is $\mathcal{G}^\omega$-closed set or the set $X - \{x\}$ is $\mathcal{G}_g^\omega$-closed set in $(X, \tau, \mathcal{G})$.

*Proof.* If $\{x\}$ is not $\mathcal{G}^\omega$-closed set in $(X, \tau, \mathcal{G})$ then $\{x\}$ is not closed set in $X$ and so $X - \{x\}$ is not open set in $X$. Then the set $X$ is only open set in itself containing $\{x\}$ and hence ${}_{\mathcal{G}^\omega}Cl(X - \{x\}) \subseteq X$. That is, $X - \{x\}$ is $\mathcal{G}_g^\omega$-closed set in $(X, \tau, \mathcal{G})$. □

**Theorem 5.9.** A subset $G$ of a grill topological space $(X, \tau, \mathcal{G})$ is $\mathcal{G}_g^\omega$-closed if and only if for each $x \in {}_{\mathcal{G}^\omega}Cl(G)$, $Cl(\{x\}) \cap G \neq \emptyset$.

*Proof.* Suppose that $G$ is $\mathcal{G}_g^\omega$-closed set in $(X, \tau, \mathcal{G})$ and $x \in {}_{\mathcal{G}^\omega}Cl(G)$ be any point. Let $Cl(\{x\}) \cap G = \emptyset$. Since $Cl(\{x\})$ is closed set in $X$ then $X - Cl(\{x\})$ is an open set in $X$. Since $G \subseteq X - Cl(\{x\})$ and $G$ is $\mathcal{G}_g^\omega$-closed set then ${}_{\mathcal{G}^\omega}Cl(G) \subseteq X - Cl(\{x\})$, but this contradiction with $x \notin X - Cl(\{x\})$. Hence $Cl(\{x\}) \cap G \neq \emptyset$.

Conversely, suppose that for each $x \in {}_{\mathcal{G}^\omega}Cl(G)$, $Cl(\{x\}) \cap G \neq \emptyset$ and $U$ be any open set in $X$ such that $G \subseteq U$. Let $x \in {}_{\mathcal{G}^\omega}Cl(G)$. Then $Cl(\{x\}) \cap G \neq \emptyset$. Then there is at least $z \in Cl(\{x\})$ and $z \in G$. Then $z \in Cl(\{x\})$ and $z \in U$. Since $U$ is an open set in $X$ then $\{x\} \cap U \neq \emptyset$. Hence $x \in U$ and so ${}_{\mathcal{G}^\omega}Cl(G) \subseteq U$. That is, $\mathcal{G}_g^\omega$-closed set. □

**Theorem 5.10.** A subset $G$ of a grill topological space $(X, \tau, \mathcal{G})$ is $\mathcal{G}_g^\omega$-open set if and only if $F \subseteq {}_{\mathcal{G}^\omega}Int(G)$ whenever $F \subseteq G$ and $F$ is closed subset of $(X, \tau)$.

*Proof.* Let $G$ be $\mathcal{G}_g^\omega$-open subset of $X$ and $F$ be a closed subset of $(X, \tau)$ such that $F \subseteq G$. Then $X - G$ is $\mathcal{G}_g^\omega$-closed set, $X - G \subseteq X - F$ and $X - F$ is an open subset of $(X, \tau)$. Hence Theorem (4.10), $X - {}_{\mathcal{G}^\omega}Int(G) = {}_{\mathcal{G}^\omega}Cl(X - G) \subseteq X - F$, that is, $F \subseteq {}_{\mathcal{G}^\omega}Int(G)$.

Conversely, suppose that $F \subseteq {}_{\mathcal{G}^\omega}Int(G)$ where $F$ is a closed subset of $(X, \tau)$ such that $F \subseteq G$. Then for any open subset $U$ of $(X, \tau)$ such that $X - G \subseteq U$, we have $X - U \subseteq G$ and $X - U \subseteq {}_{\mathcal{G}^\omega}Int(G)$. Then by Theorem (4.10), $X - {}_{\mathcal{G}^\omega}Int(G) = {}_{\mathcal{G}^\omega}Cl(X - G) \subseteq U$. Hence $X - G$ is $\mathcal{G}_g^\omega$-closed. That is, $G$ is a $\mathcal{G}_g^\omega$-open set. □

# 6 Conclusion

The notion of $\mathcal{G}^\omega$-open sets in grill topological spaces is developing to the notion of $\omega$-open sets in topological spaces. The fundamental notions in topological spaces such as the continuity, connectedness, compactness and separation axioms also may be introduced and investigated by using the class $\mathcal{G}^\omega$-open sets in grill topological spaces.

# Competing Interests

Authors have declared that no competing interests exist.

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