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The Reciprocal Generalized Inverse Gaussian Frailty with Application in Life Annuity Business

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Authors' contributions

This work was carried out in collaboration among all authors. Author WO designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors PW and JO managed the analyses of the study. Author RT managed the literature searches. All authors read and approved the final manuscript.

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Abstract

Aims: As shown in literature, several authors have adopted various individual frailty mixing distributions as a way of dealing with possible heterogeneity due to unobserved covariates in a group of insurers. This research contribution is to generalize the frailty mixing distribution to nest other classes of frailty distributions not in literature and apply the proposed distributions in valuation of life annuity business.

Methodology: A simulation study is done to assess the performance of the aforementioned models. The baseline parameters is estimated using Bayesian Inference and a better model is suggested for valuation of life annuity business.

Results: As a result of generalizing the frailty some new classes of frailty distributions are constructed such as; the Reciprocal Inverse Gaussian Frailty, the Inverse Gamma Frailty, the Harmonic Frailty and the Positive Hyperbolic Frailty.

From the simulation study, the proposed new frailty models shows that ignoring frailty leads to an underestimation of future residual lifetime since the survival curve shifts to the right when heterogeneity is accounted for. This is consistent with frailty literature.

The Reciprocal Inverse Gaussian model closely represents the Association of Kenya Insurers graduated rates with a slight increase in survival due to longevity risk.

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Conclusion: The proposed new frailty models show an increase in the insurers expected liability when unobserved heterogeneity is accounted for. This is consistent with frailty literature and thus can be applied to avoid underestimating the insurer's liability in the context of life annuity business.
The RIG model as proposed in estimating future liability by directly adjusting the AKI mortality rates shows an increase in longevity risk. The extent of heterogeneity of the insured group determines the level of risk. The RIG frailties should be considered for multivariate cases where the insureds are clustered in groups.

Keywords: Frailty model; generalized inverse Gaussian distribution; reciprocal inverse Gaussian distribution; harmonic distribution; positive hyperbolic distribution; Bayesian inference; life annuity insurance.

1 Introduction

Frailty modeling is based on mixture distributions where the population hazard is considered a mixture of measurable (e.g. health status) and unmeasured (e.g. congenital personal characteristics) risk factors affecting mortality. The frailty model is an extension of the Cox PH model. It is a random effects model which has a multiplicative effect on the hazard rate that adds additional risks based on each individual's information. The term "frailty" was introduced by Vaupel et al. [1] in a seminal paper on individual survival models. They discuss the impact of heterogeneity in individual mortality. Their findings showed that standard life-table methods overestimates current life expectancy and potential gains in life expectancy from health and safety interventions, while underestimating rates of individual aging, past progress in reducing mortality.

Standard life tables assume that the population under study is homogeneous. This means that all individuals in that study are subject under the same risk at a given age. Basic observation of medical statistics shows that individuals differ greatly see Vaupel et al. [1].

Frailty models have been adopted by several authors in insurance, for instance; Shusu et al. [2] applies frailty to quantify the extent of heterogeneity in Australian population mortality on life annuity rates and pension costs the results confirm significant heterogeneity exists. Ramona and Sherris [3] have used frailty model to quantify the impact of heterogeneity due to underwriting factors and frailty on annuity values the results showed that heterogeneity remains after underwriting and that frailty significantly impacts the fair value of both standard and underwritten annuities.

Annamaria and Pitacco [4] suggests adopting a frailty model for risk classification for life annuity portfolios. In particular, they identify risk groups within a population by assigning specific ranges of values to the frailty within each group. Pitacco E. [5] applies frailty modelling to analyze the impact of frailty, in its various interpretations, on the results of cash flows, profits, etc of life insurance and life annuity portfolios and related risk profiles.

Avanzi et al. [6] bootstrap data on Canadian pensioners' mortality to study the characteristics of its implied heterogeneity they find strong support for the Gamma frailty model. Nadine G. et al. [7] utilizes the frailty model to reflect mortality heterogeneity in optimal risk classification for substandard annuities. Their findings indicate that extended frailty risk classification can enhance the insurer's profitability.

Eriksson F and Scheike T [8] applies the additive Gamma frailty models to competing risks in related individuals.

2 Methodology

The multiplicative approach:

$h(t \mid u) = uh_0(t)$

$h_0(t)$ is the 'standard hazard function' corresponding to a 'standard individual', conventionally those with frailty $u = 1$ .The non-negative quantity u encompasses all other factors affecting mortality other than age which acts in a multiplicative manner. Individuals with $u > 1$ experience a force of mortality that is proportionally higher at all ages. Individuals with $u < 1$ experience proportionally lower mortality rates.

Frailty models without observed covariates:

This model is used when only survival data is available for the analysis, or when additional information is of no interest. I.e $h(t \mid u) = uh_0(t)$

This model is non-identifiable from survival data, since different combinations of $h_0(t)$ and frailty distributions may produce the same marginal hazard rate $h(t)$. The model becomes identifiable when the parametric structure of $h_0(t)$ is fixed and u is assumed to belong to some parametric distribution family.

Univariate frailty models:

This model accounts for heterogeneity due to unobserved risk factors for independent life times in a proportional hazard model. The variability can be split into a part that depends on observable risk factors, and is therefore theoretically predictable, and a part that is theoretically unpredictable, even when all relevant information is known. This model has been used by several authors Rocha [9] and Hougaard [10] to show that these two sources of variability can explain some unexpected results.

2.1 Model construction

Let T be the future life-time random variable with a continuous distribution. A non-negative random variable u is called "frailty" if the conditional hazard function given $U = u$ is given by; $h(t \mid u) = uh_0(t); t > 0$ where $h_0(t)$ is called the baseline age-specific hazard function for a "standard" individual. The "population" hazard corresponding to a randomly selected individual that is actually observed is given by; $E[h(t \mid u)] = \int uh_0(t) f(u)du = \bar{u} * h_0(t)$ increases less rapidly than for individuals. This is because the population becomes populated by more and more robust individuals as the frail members fail. The conditional survival function is given by; $s(t \mid u) = e^{-\int_0^t h(x|u)dx} = e^{-\int_0^t uh_0(x)dx}$

$s(t \mid u) = e^{-uH_0(t)}; t > 0$ where $H_0(t) = \int_0^t h_0(x)dx$ is the cumulative baseline hazard.

Comparing the conditional and unconditional survival functions yields $s(t \mid u) = \{s(t)\}^u$

This shows that an individual with frailty of level of 2 is twice as likely to die compared to a "standard" individual.

Since the frailty "u" is unobserved and considered random it is integrated out and thus the population is considered a mixture over "u". The univariate marginal survival function is;

$$s(t) = \int_0^\infty s(t \mid u) f(u) du = E[s(t \mid u)] = E[e^{-uH_0(t)}] = L_u(H_0(t)); t > 0 \tag{1}$$

The contribution is to construct a generalized frailty mixing distribution to nest other individual classes of frailty distributions.

2.1.1 Baseline hazard distribution

The baseline hazard used in this study is the Exponential Distribution.

The Exponential Baseline Distribution.

The density function, survival function and hazard function are;

$$f(t) = \lambda e^{-\lambda t}; t, \lambda > 0$$

$$s(t) = \int_t^\infty \lambda e^{-\lambda u} du = e^{-\lambda t}$$

$$h(t) = \frac{f(t)}{s(t)} = \lambda$$

$$H(t) = \int_0^t h(u) du = \int_0^t \lambda du = \lambda t$$

2.1.2 The generalized inverse Gaussian frailty

The Generalized Inverse Gaussian (GIG) distribution can be constructed under various parametizations. For instance, considering The Sichel's [11] parametization $\omega = \sqrt{\varphi\theta}$;

$$K_v(\omega) = \frac{1}{2} \int_0^\infty z^{v-1} e^{-\frac{\omega}{2}(z + \frac{1}{z})} dz$$

where $K_v(\omega)$ is a Bessel function of the third kind with order w and index v.

Using the transformation; $z = \sqrt{\frac{\varphi}{\theta}} x, dz = \sqrt{\frac{\varphi}{\theta}} dx$

$$K_v(\sqrt{\varphi\theta}) = \frac{1}{2}\int_0^\infty (\sqrt{\frac{\varphi}{\theta}})^v x^{v-1} e^{-\frac{\sqrt{\varphi\theta}}{2}(x\sqrt{\frac{\varphi}{\theta}}+\frac{1}{x\sqrt{\frac{\varphi}{\theta}}})} dx$$

$$1 = \frac{\int_0^\infty (\sqrt{\frac{\varphi}{\theta}})^v x^{v-1} e^{-\frac{1}{2}(\varphi x+\frac{\theta}{x})} dx}{2K_v(\sqrt{\varphi\theta})}$$

$$f(x) = \frac{(\sqrt{\frac{\varphi}{\theta}})^v x^{v-1} e^{-\frac{1}{2}(\varphi x+\frac{\theta}{x})}}{2K_v(\sqrt{\varphi\theta})}; x>0, \varphi>0, \theta>0, -\infty<v<\infty \tag{2}$$

Is the probability density function of a GIG(v,φ,θ)

The Laplace transform is

$$L_u(s) = E[e^{-su}] = \frac{\int_0^\infty u^{v-1} (\sqrt{\frac{\varphi}{\theta}})^v e^{-\frac{1}{2}[\varphi u+2su+\frac{\theta}{u}]} du}{2K_v(\sqrt{\varphi\theta})}$$

$$L_u(s) = \frac{(\sqrt{\frac{\varphi}{\theta}})^v 2K_v(\sqrt{(\varphi+2s)\theta}}{(\sqrt{\frac{\varphi+2s}{\theta}})^v 2K_v(\sqrt{\varphi\theta})} = \frac{(\sqrt{\frac{\varphi}{\varphi+2s}})^v K_v(\sqrt{(\varphi+2s)\theta})}{K_v(\sqrt{\varphi\theta})} \tag{3}$$

The marginal survival function at time $t>0$

$$S(t) = L_U(H_0(t))$$

$$S(t) = \frac{(\sqrt{\frac{\varphi}{\varphi+2(H_0(t))}})^v K_v(\sqrt{(\varphi+2H_0(t))\theta})}{K_v(\sqrt{\varphi\theta})}$$

Special cases

Case 1: Inverse Gaussian Distribution (IG):

Let $v = -\frac{1}{2}$ in equation (3)

$$L_U(s) = \frac{(\sqrt{\frac{\varphi}{\varphi+2s}})^{-\frac{1}{2}} K_{-\frac{1}{2}}(\sqrt{(\varphi+2s)\theta})}{K_{-\frac{1}{2}}(\sqrt{\varphi\theta})}$$

$$L_U(s) = \frac{(\sqrt{\frac{\varphi}{\varphi+2s}})^{-\frac{1}{2}} \sqrt{\frac{\pi}{2(\sqrt{(\varphi+2s)\theta})}} e^{-(\sqrt{(\varphi+2s)\theta})}}{\sqrt{\frac{\pi}{2\sqrt{\varphi\theta}}} e^{-\sqrt{\varphi\theta}}}$$

$$L_U(s) = e^{\sqrt{\varphi\theta}-\sqrt{(\varphi+2s)\theta}} \tag{4}$$

Substituting $\varphi = \frac{1}{\beta^2}, \theta = \mu^2$ we get

$$L_U(s) = \exp\{-\frac{\mu}{\beta}[(1+2\beta^2 s)^{1/2} - 1]\}$$

Mean = $-L'_u(0) = \mu\beta$

Variance = $L''_u(0)$ - $(L'_u(0))^2$ = $\mu\beta^3$

For identifiability reasons the mean is normalized to one. i.e. $\mu\beta = 1; \mu = \frac{1}{\beta}$ thus the variance $\delta^2 = \beta^2$

The Laplace transform is therefore

$$L_U(s) = \exp[\frac{1-(1+2s\delta^2)^{1/2}}{\delta^2}]$$

The marginal survival function at time $t > 0$

$$S(t) = L_U(H_0(t))$$
$$S(t) = \exp[\frac{1-(1+2(H_0(t))\delta^2)^{1/2}}{\delta^2}]$$

Case 2:Reciprocal Inverse Gaussian Distribution (RIG)

Let: $v = \frac{1}{2}$ in equation (3)

$$L_U(s)=\frac{(\sqrt{\frac{\varphi}{\varphi+2s}})^{\frac{1}{2}}K_{\frac{1}{2}}(\sqrt{(\varphi+2s)\theta})}{K_{\frac{1}{2}}(\sqrt{\varphi\theta})}$$

$$L_U(s)=\frac{(\sqrt{\frac{\varphi}{\varphi+2s}})^{\frac{1}{2}}\sqrt{\frac{\pi}{2(\sqrt{(\varphi+2s)\theta})}}e^{-(\sqrt{(\varphi+2s)\theta})}}{\sqrt{\frac{\pi}{2\sqrt{\varphi\theta}}}e^{-\sqrt{\varphi\theta}}}$$

$$L_U(s)=(\frac{\varphi}{2s+\varphi})^{\frac{1}{2}}e^{\sqrt{\varphi\theta}-\sqrt{(\varphi+2s)\theta}}$$

$$L_U(s)=(1+\frac{2s}{\varphi})^{-\frac{1}{2}}e^{\sqrt{\varphi\mu}\{1-\sqrt{(1+\frac{2s}{\varphi})}\}}$$

Substituting $\varphi=\frac{1}{\beta^2},\theta=\mu^2$ we get

$$L_U(s)=(1+2s\beta^2)^{-\frac{1}{2}}e^{\frac{\mu}{\beta}\{1-\sqrt{1+2s\beta^2}\}}$$

For identifiability the mean is normalized to one i.e.

$$E[U]=-L_U^{'}(s)=\{\beta^2(1+2s\beta^2)^{-\frac{3}{2}}+\mu\beta(1+2s\beta^2)^{-1}\}e^{\frac{\mu}{\beta}\{1-\sqrt{1+2s\beta^2}\}}$$

$$E[U]=-L_U{'}(0)=\beta^2+\mu\beta$$

$$\beta^2+\mu\beta=1;\mu=\frac{1-\beta^2}{\beta}$$

The Laplace becomes;

$$L_U(s)=(1+2s\beta^2)^{-\frac{1}{2}}e^{\frac{1-\beta^2}{\beta^2}\{1-\sqrt{1+2s\beta^2}\}}$$

The marginal survival function at time $t>0$

$$S(t) = L_U(H_0(t))$$

$$S(t) = (1+2\beta^2 H_0(t))^{-\frac{1}{2}} e^{\frac{1-\beta^2}{\beta^2}\{1-\sqrt{1+2\beta^2 H_0(t)}\}}$$

Case 3: Gamma Distribution:

Let $\theta = 0, v > 0, \varphi > 0$ in equation (3)

$$L_U(s) = \frac{(\sqrt{\frac{\varphi}{\varphi+2s}})^v K_v(\sqrt{(\varphi+2s)*0})}{K_v(\sqrt{\varphi*0})} = (\frac{\varphi}{\varphi+2s})^{\frac{v}{2}} = (1+\frac{2s}{\varphi})^{-\frac{v}{2}}$$

Substituting $\varphi = 2b, v = 2p$ we get

$$L_U(s) = (1+\frac{s}{b})^{-p}$$

$$Var(z) = (L''(0)-(L'(0))^2 = -p(-p-1)*(1+\frac{s}{b})^{-p-2}*(\frac{1}{b})^2 - (\frac{p}{b})^2 = \frac{p}{b^2}$$

For purposes of identifiability assume the distribution of U has mean normalized to one (p=b) and variance

$$\sigma^2 = \frac{1}{b}$$

The Laplace becomes

$$L(s) = (1+s\delta^2)^{-1/\delta^2}$$

The marginal survival function at time $t > 0$

$$S(t) = L_U(H_0(t))$$
$$S(t) = \{1+\delta^2 H_0(t)\}^{-1/\delta^2}$$

Case 4: The Levy Distribution:

This is a special case of the Inverse Gaussian distribution.

Let $\theta > 0, v = -\frac{1}{2}, \varphi = 0$ in equation (4)

$$L_U(s) = e^{\sqrt{\varphi\theta}-\sqrt{(\varphi+2s)\theta}}$$

when $\varphi = 0$ $L_U(s) = e^{-\sqrt{2s\theta}}$

For purposes of identifiability assume the distribution of *U* has mean normalized to 1. e.g. $E[U]=\theta=1$

The Laplace becomes

$$L_U(s)=e^{-\sqrt{2s}}$$

The marginal survival function at time $t>0$

$$S(t)=L_U(H_0(t))$$
$$S(t)=e^{-\sqrt{2H_0(t)}}$$

Case 5: The Harmonic Distribution

Let $\theta=an, v=0, \varphi=\dfrac{a}{n}$ in equation (3)

The Laplace becomes;

$$L_U(s)=\frac{K_0(\sqrt{2ans+a^2})}{K_0(a)}$$

For purposes of identifiability assume the distribution of *U* has mean normalized to 1.

$$E[U]=\frac{nK_1(1/n)}{K_0(a)}=1$$

The Laplace becomes

$$L_U(s)=\frac{K_0(\sqrt{2ans+a^2})}{nK_1(1/n)}$$

The marginal survival function at time $t>0$

$$S(t)=L_Z(H_0(t))$$
$$S(t)=\frac{K_0(\sqrt{2anH_0(t)+a^2})}{nK_1(1/n)}$$

Case 6: The Positive Hyperbolic Distribution

Let $\theta>0, v=1, \varphi>0$ in equation (3)

The Laplace becomes;

$$L_U(s) = \frac{\left(\sqrt{\frac{\varphi}{\varphi+2s}}\right) K_1\left(\sqrt{(\varphi+2s)\theta}\right)}{K_1(\sqrt{\varphi\theta})}$$

For purposes of identifiability assume the distribution of *U* has mean normalized to 1. e.g.

Let $\theta > 0, v = 1, \varphi > 0$ in equation (2)

$$f(u) = \sqrt{\frac{\varphi}{\theta}} \frac{e^{-1/2(\varphi u + \frac{\theta}{u})}}{2K_1(\sqrt{\varphi\theta})} \quad \text{u} > 0; \theta > 0; \varphi > 0$$

$$\mathrm{E}[\mathrm{U}^{\mathrm{r}}] = \frac{\sqrt{\frac{\varphi}{\theta}}}{2K_1(\sqrt{\varphi\theta})} \int_0^\infty u^r e^{-1/2(\varphi u + \frac{\theta}{u})} du$$

$$\mathrm{E}[\mathrm{U}^{\mathrm{r}}] = \frac{1}{K_1(\sqrt{\varphi\theta})} \frac{1}{2} \int_0^\infty \frac{\left(\sqrt{\frac{\varphi}{\theta}}\right)^{1+r}}{\left(\sqrt{\frac{\varphi}{\theta}}\right)^r} u^{1+r-1} e^{-1/2(\varphi u + \frac{\theta}{u})} du = \left(\sqrt{\frac{\theta}{\varphi}}\right)^r \frac{K_{1+r}(\sqrt{\varphi\theta})}{K_1(\sqrt{\varphi\theta})}$$

$$\therefore E[U] = \left(\sqrt{\frac{\theta}{\varphi}}\right) \frac{K_2(\sqrt{\varphi\theta})}{K_1(\sqrt{\varphi\theta})} = 1$$

The Laplace becomes

$$L_U(s) = \frac{\left(\sqrt{\frac{\varphi^2}{(\varphi+2s)\theta}}\right) K_1\left(\sqrt{(\varphi+2s)\theta}\right)}{K_2(\sqrt{\varphi\theta})}$$

The marginal survival function at time $t > 0$

$$S(t) = L_U(H_0(t))$$

$$S(t) = \frac{\left(\sqrt{\frac{\varphi^2}{(\varphi+2H(t))\theta}}\right) K_1\left(\sqrt{(\varphi+2H(t))\theta}\right)}{K_2(\sqrt{\varphi\theta})}$$

3 Results and Discussion

3.1 Simulation study

A simulation study is done to check the performance of the proposed models. A comparison is done on the survival function in the presence of different levels of frailty (heterogeneity) and without frailty (homogeneity). The baseline parameters are estimated via Bayesian analysis using Gibbs Sampler. RCODE is shown in APPENDIX.

CASE1

Gamma-Exponential Model

The Survival function without frailty: $S(t) = e^{-\lambda t}$

The Survival function with frailty: $S(t) = (1+\delta^2 \lambda t)^{-1/\delta^2}$

λ=0.05 (failure rate), δ2=5,30,60 (frailty levels) t=55:110 (age)

CASE2

Inverse Gaussian-Exponential Model

The Survival function without frailty: $S(t) = e^{-\lambda t}$

The Survival function with frailty: $S(t) = \exp\{\frac{1-(1+2\delta^2\lambda t)\hat{}0.5}{\delta^2})$

λ=0.05 (failure rate), δ2=5,30,60 (frailty levels) t=55:110 (age)

CASE3

Reciprocal Inverse Gaussian-Exponential Model

The Survival function without frailty: $S(t) = e^{-\lambda t}$

The Survival function with frailty: $S(t) = (1+2\beta^2\lambda t)\hat{}(-0.5)\exp\{\frac{1-\beta^2}{\beta^2} * (1-(1+2\beta^2\lambda t)\hat{}0.5)\}$

λ=0.05 (failure rate), β=0.1, t=55:110 (age)

CASE4

Levy-Exponential Model

The Survival function without frailty: $S(t) = e^{-\lambda t}$

The Survival function with frailty: $S(t) = e^{-(2\lambda t)^{0.5}}$

CASE5

Harmonic-Exponential Model

The Survival function without frailty: $S(t) = e^{-\lambda t}$

The Survival function with frailty: $S(t) = \frac{K_0(\sqrt{2an\lambda t + a^2})}{nK_1(1/n)}$

a=0.05, n=2.5 (shape and scale parameters)
CASE6

Positive Hyperbolic-Exponential Model

The Survival function without frailty: $S(t) = e^{-\lambda t}$

The Survival function with frailty: $S(t) = \dfrac{\left(\sqrt{\dfrac{\varphi^2}{(\varphi+2\lambda t)\theta}}\right) K_1\left(\sqrt{(\varphi+2\lambda t)\theta}\right)}{K_2(\sqrt{\varphi\theta})}$

theta=10, varphi=0.03(shape and scale parameters)

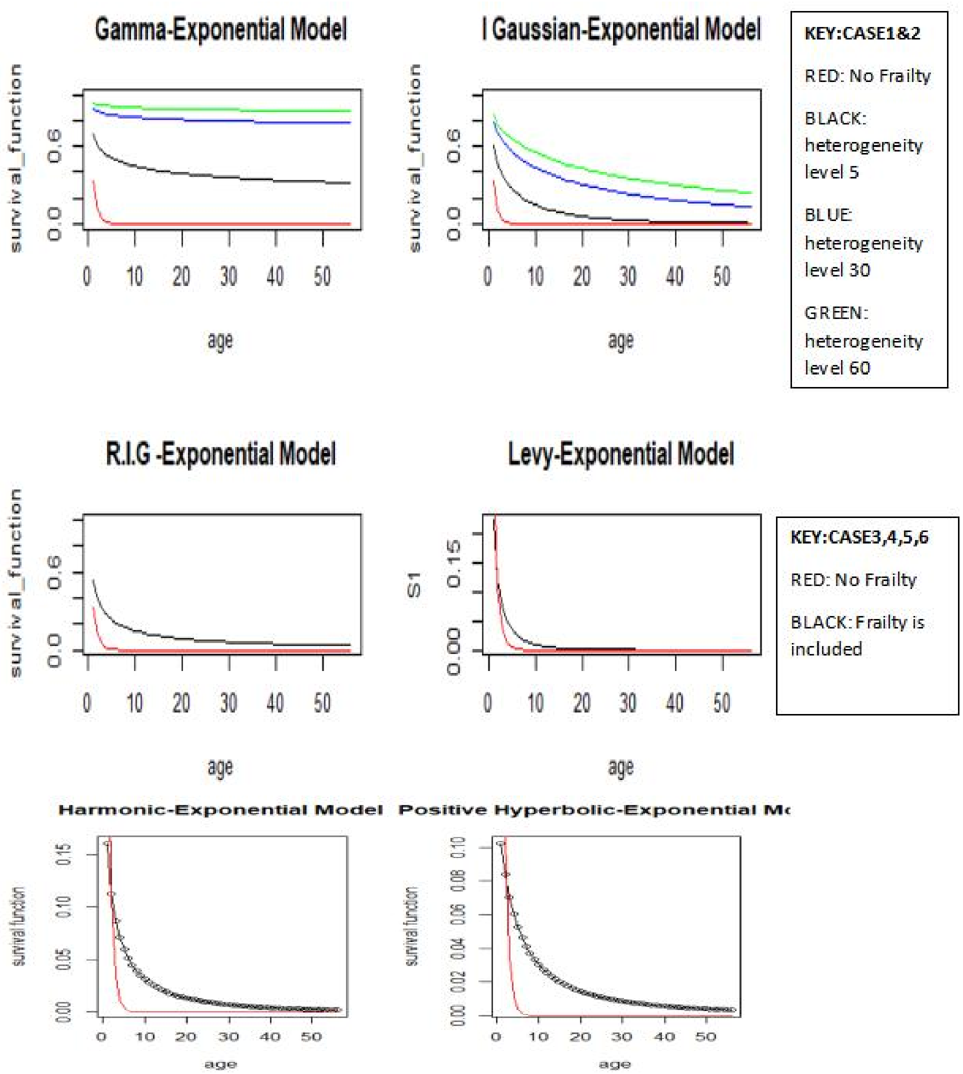

Fig. 1. Graphical representation of exponential model

Discussion:

From the simulation study above the proposed new frailty models shows that ignoring frailty leads to an underestimation of future residual lifetime since the survival curve shifts to the right when heterogeneity is accounted for this is consistent with frailty literature. The aforementioned models can therefore be applied in insurance to avoid underestimating the insurer's liability in the context of life annuity business.

Parameter estimation:

Model parameters are fixed quantitative values that characterize the model believed to reflect the real world. They have to be estimated either by statistical inference from observations or by expert opinion. Since the baseline model assumes the population to be homogeneous, the Association of Kenya Insurer's (AKI-2010) graduated rates will be considered as the baseline.

3.2 Applications in insurance industry

The pricing of long term insurance, annuity and pension products is largely influenced by the choice of the mortality projection model. Frailty models are used in life insurance to represent heterogeneity in a population due to non-observed risk factors. Heterogeneity due to observable risk factors is addressed at policy issue during the underwriting process to ensure that each contract is assigned premium consistent with the insured risk. Neglecting such factors or use of age and sex as the only rating factors (see Joelle F. [12]) may lead to mispricing of insurance products.

The aims of this exercise are:

- The first aim is to show that when heterogeneity is disregarded the expected residual lifetime is underestimated thus leading to an underestimation of the insurer's liability.
- Secondly, is to show the relevance of the proposed Reciprocal Inverse Gaussian Frailty mixture to reflect the insurer's mortality rating.

Assumption:

- The force of mortality μ is assumed piece-wise constant, taking a common value across each whole year of age $[x, x+1)$ similar assumption found in Dodd E et al. [13]
- The frailty model considered here is one without observed covariates since only survival data is available for analysis.
- Maximum age is 109
- Fixed annual interest rate of 2%

3.3 Data analysis and results

The Data:

3.3.1 The Kenyan life tables 2007-2010

The AKI 2010 table of mortality is based upon data collected by the Association of Kenya Insurers for an investigation into the mortality of assured lives in the Republic of Kenya. The method used was the census method between 2007-2010 inclusive. The AKI 2010 mortality rates will be used as the baseline hazard rate in the study.

Consider two hypothetical insurers i.e. insurer X and Y

Insurer X assumes the population to be heterogeneous with respect to underwriting factors and applies the Life-tables from the Association of Kenya Insurers (AKI 2010) graduated rates.

Insurer Y assumes the population to be heterogeneous with respect to both observable and unobserved risk factors and applies frailty modeling to modify the AKI q-rates.

The Reciprocal Inverse Gaussian-Exponential Model

$$S(t) = (1+2\beta^2 H_0(t))\hat{}(-0.5)\exp\{\frac{1-\beta^2}{\beta^2} * (1-(1+2\beta^2 H_0(t))\hat{}0.5)\}$$

Where H0(t)~AKI 2010 q-rates

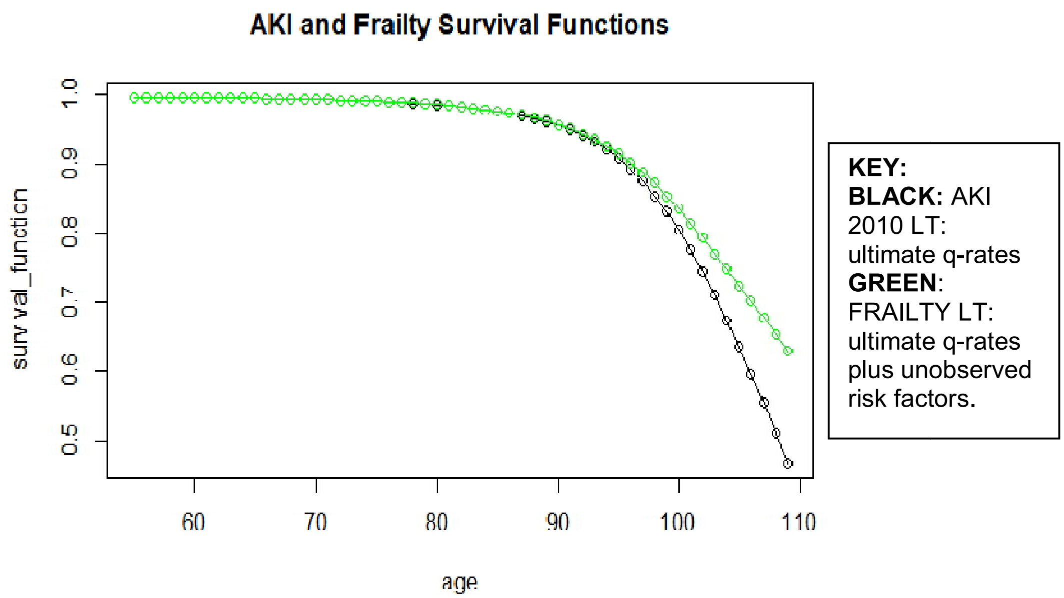

Fig. 2. AKI and frailty survival functions

Discussion:

1. Ignoring heterogeneity due to other factors affecting mortality other than age and sex ie applying the AKI 2010 rates as it is leads to an underestimation of life expectancy as the survival curve shifts to the right when heterogeneity is accounted for.
2. The Reciprocal Inverse Gaussian model closely represents the AKI 2010 graduated rates with a slight increase in survival due to longevity risk.

3.3.2 Deferred life annuity business

These are annuities which commence in m (say) years' time, provided that the annuitant is then active. Thus the present value of amount b payable for a future lifetime T_{x+t}

$$m\,|\,\bar{a}_x = \frac{D_{x+m}}{D_x} * a_{x+m}$$

Where $\frac{D_{x+m}}{D_x}$ is a pure endowment factor and a_{x+m} is an annuity factor at age x+m

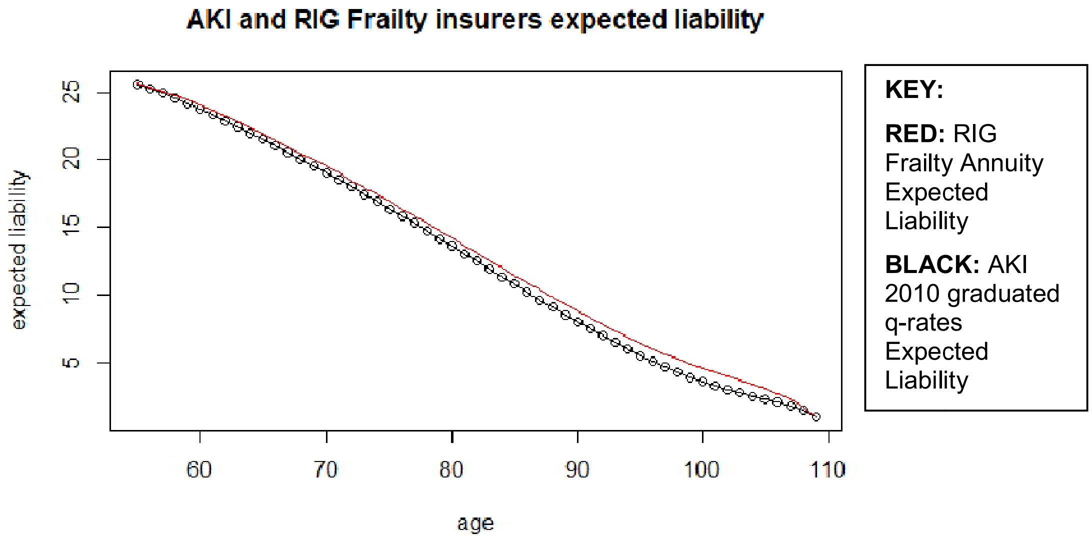

Fig. 3. AKI and RIG frailty insurers expected liability

Discussion:

1. When heterogeneity is disregarded the expected liability is underestimated.
2. The Reciprocal Inverse Gaussian frailty is a close estimate of the insurer liability with a slight increase due to observed heterogeneity.

4 Conclusion

The proposed new frailty models show an increase in the insurers expected liability when unobserved heterogeneity is accounted for. This is consistent with frailty literature and thus can be applied to avoid underestimating the insurer's liability in the context of life annuity business.

The RIG model as proposed in estimating future liability by directly adjusting the AKI mortality rates shows an increase in longevity risk. The extent of heterogeneity of the insured group determines the level of risk. The RIG frailties should be considered for multivariate cases where the insureds are clustered in groups.

Competing Interests

Authors have declared that no competing interests exist.

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APPENDIX 1A

WINBUGS CODE

```
MODEL<-FUNCTION(){
FOR(I IN 1:55){
S[I] ~ DEXP(LAMBDA)    }
LAMBDA ~ DEXP(0.09)    }
WRITE.MODEL(MODEL,"MODEL.TXT")
INIT<-FUNCTION(){LIST(ALPHA=DEXP(0.9))}
DATA=LIST(S=GIG_TABLE$S.X._AKI.MALE.I.2.)
BUGS=BUGS(DATA=DATA,INITS=INIT,PARAMETERS.TO.SAVE=C("LAMBDA"),MODEL.FILE="
MODEL.TXT",BUGS.DIRECTORY="C:/USERS/DELL/DOCUMENTS/R/WIN-
LIBRARY/3.5/R2WINBUGS",N.CHAINS=1,N.ITER=100000,N.BURNIN=100,CODAPKG=TRUE,DEB
UG=T)
```

Node Statistics

NODE	MEAN	SD	MC ERROR	2.5%	MEDIAN	97.5%	START	SAMPLE
DEVIANCE	100.8	1.396	0.04448	99.8	100.3	104.8	3	1010
LAMBDA	1.115	0.1478	0.004396	0.8378	1.105	1.422	3	1010

CASE 1 R-CODE:

```
PAR(MFROW=C(2,2))
AGE=56:1
SURVIVAL_FUNCTION=SEQ(0,1,1/55)
LAMBDA=1.115# WINBUGS ESTIMATE
S1 <-(1+5*LAMBDA*AGE)^(-1/5)
S2 <-(1+30*LAMBDA*AGE)^(-1/30)
S3 <-(1+60*LAMBDA*AGE)^(-1/60)
S4 <- EXP(-LAMBDA*AGE)
PLOT(AGE,SURVIVAL_FUNCTION,TYPE="N",MAIN="GAMMA-EXPONENTIAL MODEL")
LINES(AGE,S1,COL="BLACK")
LINES(AGE,S2,COL="BLUE")
LINES(AGE,S3,COL="GREEN")
LINES(AGE,S4,COL="RED")
```

CASE 2 R-CODE:

```
S1 <-EXP((1-(1+2*5*LAMBDA*AGE)^(0.5))/5)
S2 <-EXP((1-(1+2*30*LAMBDA*AGE)^(0.5))/30)
S3 <-EXP((1-(1+2*60*LAMBDA*AGE)^(0.5))/60)
PLOT(AGE,SURVIVAL_FUNCTION,TYPE="N",MAIN="IGAUSSIAN-EXPONENTIAL MODEL")
LINES(AGE,S1,COL="BLACK")
LINES(AGE,S2,COL="BLUE")
LINES(AGE,S3,COL="GREEN")
LINES(AGE,S4,COL="RED")
```

CASE 3 R-CODE:

```
BETA=0.9
S1<-((1+2*BETA*LAMBDA*AGE)^(-0.5))*EXP((1-BETA)*(1-
(1+2*BETA*LAMBDA*AGE)^(0.5))/BETA)
PLOT(AGE,SURVIVAL_FUNCTION,TYPE="N",MAIN="R.I.G -EXPONENTIAL MODEL")
LINES(AGE,S1,COL="BLACK")
LINES(AGE,S4,COL="RED")
```

CASE 4 R-CODE:

```
S1 <- EXP(-SQRT(2*LAMBDA*AGE))
PLOT(AGE,S1,MAIN="LEVY-EXPONENTIAL MODEL",TYPE="L")
LINES(AGE,S4,COL="RED")
```

CASE 5 R-CODE:

```
PAR(MFROW=C(1,2))
A=0.05
N=2.5
S1 <- BESSELK(SQRT(2*A*N*LAMBDA*AGE+A^2),0)/(N*BESSELK(1/N,1))
PLOT(AGE,S1,YLAB="SURVIVALFUNCTION",TYPE="O",MAIN="HARMONIC-EXPONENTIAL
MODEL")
LINES(AGE,S4,COL="RED")
```

CASE 6 R-CODE:

```
VARPHI=10
THETA=0.03
S1<-SQRT((VARPHI^2)/(VARPHI+2*LAMBDA*AGE)*THETA)*(BESSELK
(SQRT((2*LAMBDA*AGE+VARPHI)*THETA),1)/BESSELK(SQRT(THETA*VARPHI),2))
PLOT(AGE,S1,YLAB="SURVIVALFUNCTION",TYPE="O",MAIN="POSITIVE HYPERBOLIC-
EXPONENTIAL MODEL")
LINES(AGE,S4,COL="RED")
```

COMPARISON GRAPHS 1 R-CODE

```
GIG_TABLE=READ.CSV("C:/USERS/DELL/DESKTOP/PROJECTS/GIG_BOOK1.CSV",HEADER=T)
PLOT(AGE..X.,S.X._AKI.MALE.I.2.,TYPE="O",MAIN="AKI AND FRAILTY SURVIVAL
FUNCTIONS", YLAB="SURVIVAL_FUNCTION", XLAB="AGE")
BETA=0.9
RIG_SX<-((1+2*BETA*BETA*GIG_TABLE$HX_AKI)^(-0.5))*EXP((1-BETA*BETA)*(1-
(1+2*BETA*BETA*GIG_TABLE$HX_AKI)^(0.5))/(BETA*BETA))
LINES(AGE..X.,RIG_SX,COL="GREEN",TYPE="O")
```

COMPARISON GRAPHS 2 R-CODE

```
PLOT(AGE..X.,AX_AKI,TYPE="O",MAIN="AKI AND RIG FRAILTY INSURERS EXPECTED
LIABILITY", YLAB="EXPECTED LIABILITY", XLAB="AGE")
lines(Age..x.,ax_RIG,col="red")
```

Table 1. Association of Kenya Insurers Table I=2%

Age (x)	lx	dx	px	qx	µx	Dx	Nx	ax_AKI
55	100000	373	0.99627	0.00373	0.00374	33650.42	872239.81	25.92
56	99627	375	0.99624	0.00376	0.00377	32867.49	838589.39	25.51
57	99252	382	0.99615	0.00385	0.00386	32101.78	805721.90	25.10
58	98870	391	0.99604	0.00396	0.00396	31351.19	773620.12	24.68
59	98479	403	0.99591	0.00409	0.00410	30614.87	742268.93	24.25
60	98076	415	0.99577	0.00423	0.00424	29891.88	711654.06	23.81
61	97661	428	0.99562	0.00438	0.00439	29181.77	681762.18	23.36
62	97233	442	0.99545	0.00455	0.00456	28484.15	652580.41	22.91
63	96791	457	0.99528	0.00472	0.00473	27798.66	624096.26	22.45
64	96334	473	0.99509	0.00491	0.00492	27124.93	596297.59	21.98
65	95861	491	0.99488	0.00512	0.00513	26462.49	569172.66	21.51
66	95370	511	0.99464	0.00536	0.00537	25810.76	542710.17	21.03
67	94859	533	0.99438	0.00562	0.00564	25169.11	516899.41	20.54
68	94326	559	0.99408	0.00592	0.00594	24536.85	491730.29	20.04
69	93767	587	0.99375	0.00626	0.00627	23913.30	467193.44	19.54
70	93181	617	0.99337	0.00663	0.00665	23297.77	443280.14	19.03
71	92563	652	0.99296	0.00704	0.00707	22689.61	419982.37	18.51
72	91912	690	0.99250	0.00750	0.00753	22088.09	397292.76	17.99
73	91222	732	0.99198	0.00802	0.00806	21492.47	375204.67	17.46
74	90490	778	0.99140	0.00860	0.00864	20901.99	353712.21	16.92
75	89711	830	0.99075	0.00925	0.00930	20315.86	332810.22	16.38
76	88881	888	0.99001	0.00999	0.01004	19733.19	312494.36	15.84
77	87994	952	0.98918	0.01082	0.01087	19153.03	292761.17	15.29
78	87042	1023	0.98825	0.01175	0.01182	18574.39	273608.14	14.73
79	86019	1103	0.98718	0.01282	0.01290	17996.14	255033.75	14.17
80	84916	1192	0.98596	0.01404	0.01414	17417.07	237037.61	13.61
81	83724	1292	0.98456	0.01544	0.01556	16835.82	219620.54	13.04
82	82431	1405	0.98296	0.01704	0.01719	16250.92	202784.72	12.48

Age (x)	lx	dx	px	qx	µx	Dx	Nx	ax_AKI
83	81026	1531	0.98110	0.01890	0.01908	15660.76	186533.80	11.91
84	79495	1673	0.97895	0.02105	0.02127	15063.55	170873.04	11.34
85	77822	1833	0.97645	0.02356	0.02384	14457.33	155809.50	10.78
86	75989	2012	0.97352	0.02648	0.02684	13839.99	141352.17	10.21
87	73977	2213	0.97008	0.02992	0.03038	13209.28	127512.18	9.65
88	71763	2437	0.96604	0.03396	0.03455	12562.81	114302.91	9.10
89	69326	2685	0.96127	0.03873	0.03950	11898.19	101740.10	8.55
90	66641	2957	0.95562	0.04438	0.04539	11213.06	89841.91	8.01
91	63683	3252	0.94894	0.05106	0.05241	10505.34	78628.85	7.48
92	60432	3564	0.94102	0.05898	0.06079	9773.46	68123.50	6.97
93	56867	3887	0.93164	0.06836	0.07081	9016.68	58350.04	6.47
94	52980	4209	0.92056	0.07944	0.08277	8235.61	49333.36	5.99
95	48771	4510	0.90752	0.09248	0.09704	7432.75	41097.75	5.53
96	44261	4769	0.89226	0.10774	0.11400	6613.14	33665.00	5.09
97	39493	4955	0.87453	0.12548	0.13407	5784.93	27051.86	4.68
98	34537	5039	0.85411	0.14589	0.15769	4959.87	21266.92	4.29
99	29499	4989	0.83088	0.16912	0.18527	4153.23	16307.05	3.93
100	24510	4785	0.80479	0.19521	0.21718	3383.19	12153.82	3.59
101	19725	4421	0.77588	0.22412	0.25375	2669.36	8770.64	3.29
102	15305	3913	0.74432	0.25569	0.29529	2030.50	6101.27	3.00
103	11391	3300	0.71032	0.28968	0.34205	1481.70	4070.77	2.75
104	8092	2642	0.67344	0.32656	0.39536	1031.84	2589.07	2.51
105	5449	1983	0.63605	0.36395	0.45247	681.25	1557.23	2.29
106	3466	1399	0.59625	0.40375	0.51710	424.82	875.98	2.06
107	2067	920	0.55481	0.44519	0.58913	248.33	451.17	1.82
108	1147	560	0.51172	0.48828	0.66998	135.07	202.84	1.50
109	587	313	0.46686	0.53314	0.76172	67.76	67.76	1.00

Table 2. Reciprocal Inverse Gaussian frailty table

Age (x)	lx	dx	px	qx	µx	Dx	Nx	ax_RIG
55	100000	372	0.996281	0.003719	0.003726	33650	878247.8375	26.10
56	99628	374	0.996251	0.003749	0.003756	32868	844597.4126	25.70
57	99255	381	0.996162	0.003838	0.003845	32103	811729.492	25.29
58	98874	389	0.996063	0.003937	0.003945	31352	779626.8461	24.87
59	98484	401	0.995925	0.004075	0.004083	30617	748274.458	24.44
60	98083	413	0.995787	0.004213	0.004222	29894	717657.8366	24.01
61	97670	426	0.995639	0.004361	0.004371	29184	687763.8575	23.57
62	97244	440	0.995471	0.004529	0.004539	28487	658579.5088	23.12
63	96803	455	0.995303	0.004697	0.004708	27802	630092.1796	22.66
64	96349	471	0.995116	0.004884	0.004896	27129	602289.9148	22.20
65	95878	488	0.994909	0.005091	0.005104	26467	575160.8191	21.73
66	95390	508	0.994673	0.005327	0.005341	25816	548693.567	21.25
67	94882	531	0.994407	0.005593	0.005609	25175	522877.3834	20.77
68	94351	556	0.994112	0.005888	0.005905	24543	497702.2257	20.28
69	93796	583	0.993788	0.006212	0.006231	23921	473158.7423	19.78
70	93213	614	0.993416	0.006584	0.006606	23306	449238.1822	19.28
71	92599	648	0.993004	0.006996	0.007021	22698	425932.3335	18.76
72	91952	685	0.992554	0.007446	0.007474	22098	403233.8993	18.25
73	91267	727	0.992036	0.007964	0.007996	21503	381136.2169	17.72
74	90540	772	0.99147	0.00853	0.008567	20914	359633.1356	17.20
75	89768	823	0.990828	0.009172	0.009214	20329	338719.5759	16.66
76	88944	880	0.990109	0.009891	0.009940	19747	318390.9808	16.12
77	88065	942	0.989304	0.010696	0.010754	19169	298643.7836	15.58
78	87123	1012	0.988386	0.011614	0.011682	18592	279475.276	15.03
79	86111	1090	0.987344	0.012656	0.012737	18015	260883.6277	14.48
80	85021	1177	0.986153	0.013847	0.013944	17439	242868.2112	13.93
81	83844	1275	0.984794	0.015206	0.015323	16860	225429.5706	13.37
82	82569	1384	0.983241	0.016759	0.016901	16278	208569.6022	12.81

Age (x)	lx	dx	px	qx	µx	Dx	Nx	ax_RIG
83	81185	1506	0.981449	0.018551	0.018725	15691	192291.5672	12.25
84	79679	1643	0.979385	0.020615	0.020830	15098	176600.1638	11.70
85	78036	1796	0.97698	0.02302	0.023289	14497	161501.8186	11.14
86	76240	1967	0.974194	0.025806	0.026145	13886	147004.6688	10.59
87	74273	2159	0.970938	0.029062	0.029493	13262	133118.9576	10.04
88	72114	2369	0.967143	0.032857	0.033409	12624	119856.8237	9.49
89	69745	2602	0.962696	0.037304	0.038018	11970	107232.5984	8.96
90	67143	2855	0.957484	0.042516	0.043446	11298	95262.56797	8.43
91	64288	3126	0.951381	0.048619	0.049841	10605	83965.01852	7.92
92	61163	3410	0.944245	0.055755	0.057370	9892	73359.89808	7.42
93	57752	3701	0.935921	0.064079	0.066224	9157	63468.22153	6.93
94	54052	3985	0.926268	0.073732	0.076592	8402	54311.19591	6.46
95	50066	4249	0.915131	0.084869	0.088688	7630	45908.98751	6.02
96	45817	4472	0.902404	0.097596	0.102693	6846	38278.89263	5.59
97	41346	4630	0.888007	0.111993	0.118776	6056	31433.26875	5.19
98	36715	4703	0.871916	0.128084	0.137062	5273	25376.8782	4.81
99	32013	4668	0.854189	0.145811	0.157603	4507	20104.21426	4.46
100	27345	4513	0.83497	0.16503	0.180359	3774	15597.03775	4.13
101	22832	4236	0.814463	0.185537	0.205226	3090	11822.54697	3.83
102	18596	3851	0.792907	0.207093	0.232049	2467	8732.756213	3.54
103	14745	3383	0.770577	0.229423	0.260616	1918	6265.5795	3.27
104	11362	2871	0.747285	0.252715	0.291309	1449	4347.695494	3.00
105	8491	2339	0.724535	0.275465	0.322225	1062	2898.796177	2.73
106	6152	1839	0.701138	0.298862	0.355051	754	1837.285661	2.44
107	4313	1391	0.677556	0.322444	0.389263	518	1083.264561	2.09
108	2922	1012	0.653726	0.346274	0.425067	344	564.9578495	1.64
109	1910	708	0.629511	0.370489	0.462812	221	220.6619453	1.00

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