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Article

Some integral inequalities for co-ordinated harmonically convex functions via fractional integrals

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Abstract: In this paper, we find some Hermite-Hadamard type inequalities for co-ordinated harmonically convex functions via fractional integrals.

Keywords: Hermite-Hadamard inequalities, Riemann-Liouville fractional integral, co-ordinated convex functions, co-ordinated harmonically convex functions.

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1. Introduction and Preliminaries

For a convex mapping $\prod : I \to \mathbb{R}$ on a real interval, for all $f_1, f_2 \in I$ and $t \in [0,1]$, the inequality

$$\prod\left(\frac{f_1+f_2}{2}\right) \leq \frac{1}{f_2-f_1}\int_{f_1}^{f_2}\prod(u)du \leq \frac{\prod(f_1)+\prod(f_2)}{2}, \tag{1}$$

is known as the Hermite-Hadamard inequality [1]. The inequality (1) has been established for several generalized convex functions [2–9]. Dragomir [10] and Sarikaya [11] calculated Hermite-Hadamard inequality for co-ordinated convex functions. They define co-ordinated convex function as:

Definition 1. [10] A function $\prod : \Delta = [f_1, f_2] \times [g_1, g_2] \subseteq \mathbb{R}^2 \to \mathbb{R}$ is called co-ordinate convex on Δ with $f_1 < f_2$ and $g_1 < g_2$, if the partial functions

$$\prod_y : [f_1, f_2] \to \mathbb{R}, \prod_y(u) = \prod(u, y), \text{ and } \prod_x : [g_1, g_2] \to \mathbb{R}, \prod_x(v) = \prod(x, v),$$

are convex for all $x \in [f_1, f_2]$ and $y \in [g_1, g_2]$.

Sarikaya [11] define the co-ordinated convex function as:

Definition 2. [11] A function $\prod : \Delta = [f_1, f_2] \times [g_1, g_2] \subseteq \mathbb{R}^2 \to \mathbb{R}$ is called coordinate convex on Δ with $f_1 < f_2$ and $g_1 < g_2$, if

$$\begin{aligned}&\prod(t_1x+(1-t_1)z, t_2y+(1-t_2)w)\\ &\leq t_1t_2\prod(x,y)+t_1(1-t_2)\prod(x,w)+(1-t_1)t_2\prod(z,y)+(1-t_1)(1-t_2)\prod(z,w),\end{aligned}$$

holds for all $t_1, t_2 \in [0,1]$ and $(x,y), (z,w) \in \Delta$.

Every convex function is co-ordinated convex but not conversely [10].

Theorem 3. *[10] Let $\prod : \Delta = [f_1, f_2] \times [g_1, g_2] \subseteq \mathbb{R}^2 \to \mathbb{R}$ be convex on Δ with $f_1 < f_2$ and $g_1 < g_2$. Then*

$$\begin{aligned}\prod\left(\frac{f_1+f_2}{2}, \frac{g_1+g_2}{2}\right) &\leq \frac{1}{2}\left[\frac{1}{f_2-f_1}\int_{f_1}^{f_2}\prod\left(x, \frac{g_1+g_2}{2}\right)dx + \frac{1}{g_2-g_1}\int_{g_1}^{g_2}\prod\left(\frac{f_1+f_2}{2}, y\right)dy\right]\\ &\leq \frac{1}{(f_2-f_1)(g_2-g_1)}\int_{g_1}^{g_2}\int_{f_1}^{f_2}\prod(x,y)dxdy\end{aligned}$$

$$\leq \frac{1}{4}\left[\frac{1}{f_2-f_1}\int_{f_1}^{f_2}\prod(x,g_1)dx+\frac{1}{f_2-f_1}\int_{f_1}^{f_2}\prod(x,d)dx\right.$$
$$\left.+\frac{1}{g_2-g_1}\int_{g_1}^{g_2}\prod(f_1,y)dy+\frac{1}{g_2-g_1}\int_{g_1}^{g_2}\prod(f_2,y)dy\right]$$
$$\leq \frac{\prod(f_1,g_2)+\prod(f_1,g_2)+\prod(f_2,g_1)+\prod(f_2,g_2)}{4}. \tag{2}$$

Definition 4. [12] A function $\prod : \Delta = [f_1, f_2] \times [g_1, g_2] \subseteq \mathbb{R}^2 \to \mathbb{R}$ is called harmonically convex on Δ with $f_1 < f_2$ and $g_1 < g_2$, if

$$\prod\left(\frac{xz}{t_1x+(1-t_1)z},\frac{yw}{t_2y+(1-t_2)w}\right) \leq t_1t_2\prod(x,y)+(1-t_1)(1-t_2)\prod(z,w),$$

holds for all $t_1, t_2 \in [0, 1]$ and $(x, y), (z, w) \in \Delta$.

Definition 5. [12] A function $\prod : \Delta = [f_1, f_2] \times [g_1, g_2] \subseteq (0,\infty) \times (0,\infty) \to \mathbb{R}$ is called coordinated harmonically convex on Δ with $f_1 < f_2$ and $g_1 < g_2$, if

$$\prod\left(\frac{xz}{t_1x+(1-t_1)z},\frac{yw}{t_2y+(1-t_2)w}\right)$$
$$\leq t_1t_2\prod(x,y)+t_1(1-t_2)\prod(x,w)+(1-t_1)t_2\prod(z,y)+(1-t_1)(1-t_2)\prod(z,w),$$

holds for all $t_1, t_2 \in [0, 1]$ and $(x, y), (z, w) \in \Delta$.

Note that, a function $\prod : \Delta = [f_1, f_2] \times [g_1, g_2] \subseteq (0,\infty) \times (0,\infty) \to \mathbb{R}$ is called coordinated harmonically convex on Δ with $f_1 < f_2$ and $g_1 < g_2$, if the partial functions

$$\prod_y : [f_1, f_2] \to \mathbb{R}, \prod_y(u) = \prod(u, y), \prod_x : [g_1, g_2] \to \mathbb{R}, \prod_x(v) = \prod(x, v),$$

are harmonically convex for all $x \in [f_1, f_2]$ and $y \in [g_1, g_2]$, (for more detail, see [9,12]).

Theorem 6. *[12] Let $\prod : \Delta = [f_1, f_2] \times [g_1, g_2] \subseteq (0,\infty) \times (0,\infty) \to \mathbb{R}$ be co-ordinated harmonically convex on Δ with $f_1 < f_2$ and $g_1 < g_2$. Then*

$$\prod\left(\frac{2f_1f_2}{f_1+f_2},\frac{2g_1g_2}{g_1+g_2}\right) \leq \frac{(f_1f_2)(g_1g_2)}{(f_2-f_1)(g_2-g_1)}\int_{f_1}^{f_2}\int_{g_1}^{g_2}\frac{\prod(x,y)}{x^2y^2}dydx$$
$$\leq \frac{\prod(f_1,g_1)+\prod(f_1,g_2)+\prod(f_2,g_1)+\prod(f_2,g_2)}{4}. \tag{3}$$

Definition 7. [13] Let $\prod \in L[f_1, f_2]$. The right-hand side and left-hand side Riemann- Liouville fractional integrals $J^{\alpha}_{f_1+}\prod$ and $J^{\alpha}_{f_2-}\prod$ of order $\alpha > 0$ with $f_2 > f_1 \geq 0$ are defined by

$$J^{\alpha}_{f_1+}\prod(x) = \frac{1}{\Gamma(\alpha)}\int_{f_1}^{x}(x-t)^{\alpha-1}\prod(t)dt,\ x > f_1,$$

and

$$J^{\alpha}_{f_2-}\prod(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{f_2}(t-x)^{\alpha-1}\prod(t)dt,\ x < f_2,$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}dt$.

Theorem 8. *[14] Let $\prod : I \subseteq (0,\infty) \to \mathbb{R}$ be a function such that $\prod \in L_1(f_1, f_2)$ where $f_1, f_2 \in I$ with $f_1 < f_2$. If $\prod$ is harmonocally convex function on $[f_1, f_2]$, then following inequality for fractional integral hold:*

$$\prod\left(\frac{2f_1f_2}{f_1+f_2}\right) \leq \frac{\Gamma(\alpha+1)}{2}\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}\left[J^{\alpha}_{1/f_1-}\left(\prod\circ\Omega\right)\left(\frac{1}{f_2}\right)+J^{\alpha}_{1/f_2+}\left(\prod\circ\Omega\right)\left(\frac{1}{f_1}\right)\right] \leq \frac{\prod(f_1)+\prod(f_2)}{2}, \tag{4}$$

where $\alpha > 0$ *and* $\Omega(x) = \frac{1}{x}$.

Definition 9. [11] Let $\prod \in L_1([f_1, f_2] \times [g_1, g_2])$. The Riemann-Liouville integrals $J^{\alpha,\beta}_{f_1+,g_1+}$, $J^{\alpha,\beta}_{f_1+,g_2-}$, $J^{\alpha,\beta}_{f_2-,g_1+}$ and $J^{\alpha,\beta}_{f_2-,g_2-}$ of order $\alpha, \beta > 0$ with $f_1, g_1 \geq 0$ are defined by

$$J^{\alpha,\beta}_{f_1+,g_1+}\prod(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{f_1}^{x}\int_{g_1}^{y}(x-t)^{\alpha-1}(y-s)^{\beta-1}\prod(t,s)dsdt,\ x > f_1\ y > g_1,$$

$$J^{\alpha,\beta}_{f_1+,g_2-}\prod(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{f_1}^{x}\int_{y}^{g_2}(x-t)^{\alpha-1}(y-s)^{\beta-1}\prod(t,s)dsdt,\ x > f_1\ y < g_2,$$

$$J^{\alpha,\beta}_{f_2-,g_1+}\prod(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{x}^{f_2}\int_{g_1}^{y}(x-t)^{\alpha-1}(y-s)^{\beta-1}\prod(t,s)dsdt,\ x < f_2\ y > g_1,$$

and

$$J^{\alpha,\beta}_{f_2-,g_2-}\prod(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{x}^{f_2}\int_{y}^{g_2}(x-t)^{\alpha-1}(y-s)^{\beta-1}\prod(t,s)dsdt,\ x < f_2\ y < g_2,$$

respectively. Here Γ is the Gamma function.

Theorem 10. *[11] Let* $\prod : \Delta = [f_1, f_2] \times [g_1, g_2] \subseteq \mathbb{R}^2 \to \mathbb{R}$ *be convex on* Δ *with* $f_1 < f_2$ *and* $g_1 < g_2$ *and* $\prod \in L_1(\Delta)$. *Then*

$$\begin{aligned}\prod\left(\frac{f_1+f_2}{2}, \frac{g_1+g_2}{2}\right) &\leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(f_2-f_1)^{\alpha}(g_2-g_1)^{\beta}}\\ &\times\left[J^{\alpha,\beta}_{f_1+,g_1+}\prod(f_2,g_2) + J^{\alpha,\beta}_{f_1+,g_2-}\prod(f_2,g_1) + J^{\alpha,\beta}_{f_2-,g_1+}\prod(f_1,g_2) + J^{\alpha,\beta}_{f_2-,g_2-}\prod(f_1,g_1)\right]\\ &\leq \frac{\prod(f_1,g_1)+\prod(f_1,g_2)+\prod(f_2,g_1)+\prod(f_2,g_2)}{4}. \end{aligned} \tag{5}$$

In this paper, we gave integral results for co-ordinated harmonically convex functions via fractional integrals.

2. Main Results

In this section, our aim is to prove some Hermite-Hadamard type ineqalities for co-ordinated harmonically convex functions in fractional integrals.

Theorem 11. *Let* $\prod : \Delta = [f_1, f_2] \times [g_1, g_2] \subseteq (0,\infty) \times (0,\infty) \to \mathbb{R}$ *be harmonically convex on* Δ *with* $f_1 < f_2$ *and* $g_1 < g_2$ *and* $\prod \in L_1(\Delta)$. *Then*

$$\begin{aligned}\prod\left(\frac{2f_1f_2}{f_1+f_2}, \frac{2g_1g_2}{g_1+g_2}\right) &\leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4}\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}\left(\frac{g_1g_2}{g_2-g_1}\right)^{\beta}\\ &\times\left[J^{\alpha,\beta}_{1/f_1-,1/g_1-}(\textstyle\prod\circ\Omega)\left(\frac{1}{f_2}, \frac{1}{g_2}\right) + J^{\alpha,\beta}_{1/f_1-,1/g_2+}(\textstyle\prod\circ\Omega)\left(\frac{1}{f_2}, \frac{1}{g_1}\right)\right.\\ &\left.+ J^{\alpha,\beta}_{1/f_2+,1/g_1-}(\textstyle\prod\circ\Omega)\left(\frac{1}{f_1}, \frac{1}{g_2}\right) + J^{\alpha,\beta}_{1/f_2+,1/g_2+}(\textstyle\prod\circ\Omega)\left(\frac{1}{f_1}, \frac{1}{g_1}\right)\right]\\ &\leq \frac{\prod(f_1,g_2)+\prod(f_1,g_2)+\prod(f_2,g_1)+\prod(f_2,g_2)}{4}, \end{aligned} \tag{6}$$

where $\Omega(x,y) = \left(\frac{1}{x}, \frac{1}{y}\right)$ *for all* $(x,y) \in ([\frac{1}{f_2}, \frac{1}{f_1}], [\frac{1}{g_2}, \frac{1}{g_1}])$.

Proof. Let $(x,y), (z,w) \in \Delta$ and $t_1, t_2 \in [0,1]$. Since $\prod$ is co-ordinated harmonically convex on Δ, we have

$$\prod\left(\frac{xz}{t_1x+(1-t_1)z},\frac{yw}{t_2y+(1-t_2)w}\right)$$
$$\leq t_1t_2\prod(x,y)+t_1(1-t_2)\prod(x,w)+(1-t_1)t_2\prod(z,y)+(1-t_1)(1-t_2)\prod(z,w). \tag{7}$$

By taking $x=\frac{f_1f_2}{t_1f_1+(1-t_1)f_2}$, $z=\frac{f_1f_2}{t_1f_2+(1-t_1)f_1}$, $y=\frac{g_1g_2}{t_2g_1+(1-t_2)g_2}$, $w=\frac{g_1g_2}{t_2g_2+(1-t_2)g_1}$ and $t_1=t_2=\frac{1}{2}$ in (7), we get

$$\prod\left(\frac{2f_1f_2}{f_1+f_2},\frac{2g_1g_2}{g_1+g_2}\right)$$
$$\leq\frac{1}{4}\Bigg[\prod\left(\frac{f_1f_2}{t_1f_1+(1-t_1)f_2},\frac{g_1g_2}{t_2g_1+(1-t_2)g_2}\right)+\prod\left(\frac{f_1f_2}{t_1f_1+(1-t_1)f_2},\frac{g_1g_2}{t_2g_2+(1-t_2)g_1}\right)$$
$$+\prod\left(\frac{f_1f_2}{t_1f_2+(1-t_1)f_1},\frac{g_1g_2}{t_2g_2+(1-t_2)g_1}\right)+\prod\left(\frac{f_1f_2}{t_1f_2+(1-t_1)f_1},\frac{g_1g_2}{t_2g_1+(1-t_2)g_2}\right)\Bigg]. \tag{8}$$

Multiplying both sides of (8) by $t_1^{\alpha-1}t_2^{\beta-1}$ and then integrating with respect to (t_1,t_2) over $[0,1]\times[0,1]$, we get

$$\frac{1}{\alpha\beta}\prod\left(\frac{2f_1f_2}{f_1+f_2},\frac{2g_1g_2}{g_1+g_2}\right)\leq\frac{1}{4}\Bigg[\int_0^1\int_0^1\Bigg\{\prod\left(\frac{f_1f_2}{t_1f_1+(1-t_1)f_2},\frac{g_1g_2}{t_2g_1+(1-t_2)g_2}\right)$$
$$+\prod\left(\frac{f_1f_2}{t_1f_1+(1-t_1)f_2},\frac{g_1g_2}{t_2g_2+(1-t_2)g_1}\right)\Bigg\}t_1^{\alpha-1}t_2^{\beta-1}dt_1dt_2$$
$$+\int_0^1\int_0^1\Bigg\{\prod\left(\frac{f_1f_2}{t_1f_2+(1-t_1)f_1},\frac{g_1g_2}{t_2g_1+(1-t_2)g_2}\right)$$
$$+\prod\left(\frac{f_1f_2}{t_1f_2+(1-t_1)f_1},\frac{g_1g_2}{t_2g_2+(1-t_2)g_1}\right)\Bigg\}t_1^{\alpha-1}t_2^{\beta-1}dt_1dt_2\Bigg]. \tag{9}$$

Applying change of variable, we find

$$\prod\left(\frac{2f_1f_2}{f_1+f_2},\frac{2g_1g_2}{g_1+g_2}\right)\leq\frac{\alpha\beta}{4}\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}\left(\frac{g_1g_2}{g_2-g_1}\right)^{\beta}$$
$$\times\Bigg[\int_{1/g_2}^{1/g_1}\int_{1/f_2}^{1/f_1}\Bigg\{\left(\frac{1}{f_1}-x\right)^{\alpha-1}\left(\frac{1}{g_1}-y\right)^{\beta-1}\prod\left(\frac{1}{x},\frac{1}{y}\right)+\left(\frac{1}{f_1}-x\right)^{\alpha-1}\left(y-\frac{1}{g_2}\right)^{\beta-1}\prod\left(\frac{1}{x},\frac{1}{y}\right)\Bigg\}dxdy$$
$$+\int_{1/g_2}^{1/g_1}\int_{1/f_2}^{1/f_1}\Bigg\{\left(x-\frac{1}{f_2}\right)^{\alpha-1}\left(\frac{1}{g_1}-y\right)^{\beta-1}\prod\left(\frac{1}{x},\frac{1}{y}\right)+\left(x-\frac{1}{f_2}\right)^{\alpha-1}\left(y-\frac{1}{g_2}\right)^{\beta-1}\prod\left(\frac{1}{x},\frac{1}{y}\right)\Bigg\}dxdy\Bigg]. \tag{10}$$

Then by multiplying and dividing by $\Gamma(\alpha)\Gamma(\beta)$ on right hand side of inequality (10), we get the first inequality of (6). For the second inequality of (6) we use the co-ordinated harmonically convexity of $\prod$ as:

$$\prod\left(\frac{f_1f_2}{t_1f_1+(1-t_1)f_2},\frac{g_1g_2}{t_2g_1+(1-t_2)g_2}\right)$$
$$\leq t_1t_2\prod(f_1,g_1)+t_1(1-t_2)\prod(f_1,g_2)+(1-t_1)t_2\prod(f_2,g_1)+(1-t_1)(1-t_2)\prod(f_2,g_2),$$

$$\prod\left(\frac{f_1f_2}{t_1f_1+(1-t_1)f_2},\frac{g_1g_2}{t_2g_2+(1-t_2)g_1}\right)$$
$$\leq t_1t_2\prod(f_1,g_2)+t_1(1-t_2)\prod(f_1,g_1)+(1-t_1)t_2\prod(f_2,g_2)+(1-t_1)(1-t_2)\prod(f_2,g_1),$$

$$\prod\left(\frac{f_1f_2}{t_1f_2+(1-t_1)f_1},\frac{g_1g_2}{t_2g_1+(1-t_2)g_2}\right)$$
$$\le t_1t_2\prod(f_2,g_1)+t_1(1-t_2)\prod(f_2,g_2)+(1-t_1)t_2\prod(f_1,g_1)+(1-t_1)(1-t_2)\prod(f_1,g_2),$$

and

$$\prod\left(\frac{f_1f_2}{t_1f_2+(1-t_1)f_1},\frac{g_1g_2}{t_2g_2+(1-t_2)g_1}\right)$$
$$\le t_1t_2\prod(f_2,g_2)+t_1(1-t_2)\prod(f_2,g_1)+(1-t_1)t_2\prod(f_1,g_2)+(1-t_1)(1-t_2)\prod(f_1,g_1).$$

Then by adding above inequalities, we get

$$\prod\left(\frac{f_1f_2}{t_1f_1+(1-t_1)f_2},\frac{g_1g_2}{t_2g_1+(1-t_2)g_2}\right)+\prod\left(\frac{f_1f_2}{t_1f_1+(1-t_1)f_2},\frac{g_1g_2}{t_2g_2+(1-t_2)g_1}\right)$$
$$+\prod\left(\frac{f_1f_2}{t_1f_2+(1-t_1)f_1},\frac{g_1g_2}{t_2g_1+(1-t_2)g_2}\right)+\prod\left(\frac{f_1f_2}{t_1f_2+(1-t_1)f_1},\frac{g_1g_2}{t_2g_2+(1-t_2)g_1}\right)$$
$$\le\prod(f_1,g_1)+\prod(f_2,g_1)+\prod(f_1,g_2)+\prod(f_2,g_2). \tag{11}$$

Thus by multiplying (11) by $t_1^{\alpha-1}t_2^{\beta-1}$ and then integrating with respect to (t_1,t_2) over $[0,1]\times[0,1]$, we get the second inequality of (6). Hence the proof is completed.

□

Remark 1. In Theorem 11, if one takes $\alpha=\beta=1$ and using change of variable $u=1/x$ and $v=1/y$, then one has Theorem in [12].

Theorem 12. *Let $\prod:\Delta=[f_1,f_2]\times[g_1,g_2]\subseteq(0,\infty)\times(0,\infty)\to\mathbb{R}$ be harmonically convex on Δ with $f_1<f_2$ and $g_1<g_2$ and $\Psi\in L_1(\Delta)$. Then*

$$\prod\left(\frac{2f_1f_2}{f_1+f_2},\frac{2g_1g_2}{g_1+g_2}\right)\le\frac{\Gamma(\alpha+1)}{4}\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}$$
$$\times\left[J^{\alpha}_{1/f_2+}(\prod\circ\Omega_1)\left(\frac{1}{f_1},\frac{2g_1g_2}{g_1+g_2}\right)+J^{\alpha}_{1/c_1-}(\prod\circ\Omega_1)\left(\frac{1}{f_2},\frac{2g_1g_2}{g_1+g_2}\right)\right]+\frac{\Gamma(\beta+1)}{4}\left(\frac{g_1g_2}{g_2-g_1}\right)^{\beta}$$
$$\times\left[J^{\beta}_{1/g_2+}(\prod\circ\Omega_2)\left(\frac{2f_1f_2}{f_1+f_2},\frac{1}{g_1}\right)+J^{\beta}_{1/g_1-}(\prod\circ\Omega_2)\left(\frac{2f_1f_2}{f_1+f_2},\frac{1}{g_2}\right)\right]$$
$$\le\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{2}\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}\left(\frac{g_1g_2}{g_2-g_1}\right)^{\beta}\times\left[J^{\alpha,\beta}_{f_1+,g_1+}(\prod\circ\Omega)\left(\frac{1}{f_2},\frac{1}{g_1}\right)\right.$$
$$\left.+J^{\alpha,\beta}_{f_1+,g_2-}(\prod\circ\Omega)\left(\frac{1}{f_2},\frac{1}{g_1}\right)+J^{\alpha,\beta}_{f_2-,g_1+}(\prod\circ\Omega)\left(\frac{1}{f_1},\frac{1}{g_2}\right)+J^{\alpha,\beta}_{f_2-,g_2-}(\prod\circ\Omega)\left(\frac{1}{f_1},\frac{1}{g_1}\right)\right]$$
$$\le\frac{\Gamma(\alpha+1)}{4}\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}\left[J^{\alpha}_{1/f_2+}(\prod\circ\Omega_1)\left(\frac{1}{f_1},g_2\right)+J^{\alpha}_{1/f_2+}(\prod\circ\Omega_1)\left(\frac{1}{f_1},g_1\right)\right.$$
$$\left.+J^{\alpha}_{1/f_1-}(\prod\circ\Omega_1)\left(\frac{1}{f_2},g_1\right)+J^{\alpha}_{1/f_1-}(\prod\circ\Omega_1)\left(\frac{1}{f_2},g_1\right)\right]$$
$$+\frac{\Gamma(\beta+1)}{4}\left(\frac{g_1g_2}{g_2-g_1}\right)^{\alpha}\left[J^{\beta}_{1/g_1-}(\prod\circ\Omega_2)\left(f_1,\frac{1}{g_2}\right)+J^{\beta}_{1/g_1-}(\prod\circ\Omega_2)\left(f_2,\frac{1}{g_2}\right)\right.$$
$$\left.+J^{\alpha}_{1/g_2+}(\prod\circ\Omega_2)\left(f_1,\frac{1}{g_1}\right)+J^{\alpha}_{1/g_2+}(\prod\circ\Omega_2)\left(f_2,\frac{1}{g_1}\right)\right]$$
$$\le\frac{\Pi(f_1,g_1)+\Pi(f_1,g_2)+\Pi(f_2,g_1)+\Pi(f_2,g_2)}{4}, \tag{12}$$

where $\Omega(x,y)=\left(\frac{1}{x},\frac{1}{y}\right)$, $\Omega_1(x,y)=\left(\frac{1}{x},y\right)$ and $\Omega_2(x,y)=\left(x,\frac{1}{y}\right)$ for all $(x,y)\in\left([\frac{1}{f_2},\frac{1}{f_1}],[\frac{1}{g_2},\frac{1}{g_1}]\right)$.

Proof. Since $\prod$ is co-ordinated harmonically convex on Δ then we have $\prod_{\frac{1}{x}} : [f_1, f_2] \to \mathbb{R}$, $\prod_{\frac{1}{x}}(y) = \prod(\frac{1}{x}, y)$, is harmonically convex on $[g_1, g_2]$ for all $x \in \left[\frac{1}{f_2}, \frac{1}{f_1}\right]$. Then from inequality (4), we have

$$\prod_{\frac{1}{x}}\left(\frac{2g_1g_2}{g_1+g_2}\right) \leq \frac{\Gamma(\beta+1)}{2}\left(\frac{g_1g_2}{g_2-g_1}\right)^{\beta}\left[J^{\beta}_{1/c-}(\prod_{\frac{1}{x}}\circ\Omega_2)\left(\frac{1}{g_2}\right) + J^{\beta}_{1/g_2+}(\prod_{\frac{1}{x}}\circ\Omega_2)\left(\frac{1}{g_1}\right)\right]$$
$$\leq \frac{\prod_{\frac{1}{x}}(g_1) + \prod_{\frac{1}{x}}(g_2)}{2}. \tag{13}$$

In other words,

$$\prod\left(\frac{1}{x}, \frac{2g_1g_2}{g_1+g_2}\right) \leq \frac{\beta}{2}\left(\frac{g_1g_2}{g_2-g_1}\right)^{\beta}\left[\int_{1/g_2}^{1/g_1}\left(y-\frac{1}{g_2}\right)^{\beta-1}\prod\left(\frac{1}{x}, \frac{1}{y}\right)\mathrm{d}y\right.$$
$$\left.+\int_{1/g_2}^{1/g_1}\left(\frac{1}{g_1}-y\right)^{\beta-1}\prod\left(\frac{1}{x}, \frac{1}{y}\right)\mathrm{d}y\right] \leq \frac{\prod\left(\frac{1}{x}, g_1\right) + \prod\left(\frac{1}{x}, g_2\right)}{2}, \tag{14}$$

for all $x \in \left[\frac{1}{f_2}, \frac{1}{f_1}\right]$. Now by multiplying (14) by $\frac{\alpha(x-1/f_2)^{\alpha-1}}{2}\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}$ and $\frac{\alpha(1/f_1-x)^{\alpha-1}}{2}\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}$, and then integrating with respect to x over $[1/f_2, 1/f_1]$, respectively, we find

$$\frac{\alpha}{2}\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}\int_{1/f_2}^{1/f_1}\left(x-\frac{1}{f_2}\right)^{\alpha-1}\prod\left(\frac{1}{x}, \frac{2g_1g_2}{g_1+g_2}\right)\mathrm{d}x \leq \frac{\alpha\beta}{4}\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}\left(\frac{g_1g_2}{g_2-g_1}\right)^{\beta}$$
$$\times\left[\int_{1/f_2}^{1/f_1}\int_{1/g_2}^{1/g_1}\left(x-\frac{1}{f_2}\right)^{\alpha-1}\left(y-\frac{1}{g_2}\right)^{\beta-1}\prod\left(\frac{1}{x}, \frac{1}{y}\right)\mathrm{d}y\mathrm{d}x\right.$$
$$\left.+\int_{1/f_2}^{1/f_1}\int_{1/g_2}^{1/g_1}\left(x-\frac{1}{f_2}\right)^{\alpha-1}\left(\frac{1}{g_1}-y\right)^{\beta-1}\prod\left(\frac{1}{x}, \frac{1}{y}\right)\mathrm{d}y\mathrm{d}x\right]$$
$$\leq \frac{\alpha\beta}{4}\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}\left[\int_{1/f_2}^{1/f_1}\left(x-\frac{1}{f_2}\right)^{\alpha-1}\prod\left(\frac{1}{x}, g_1\right)\mathrm{d}x + \int_{1/f_2}^{1/f_1}\left(x-\frac{1}{f_2}\right)^{\alpha-1}\prod\left(\frac{1}{x}, g_2\right)\mathrm{d}x\right], \tag{15}$$

and

$$\frac{\alpha}{2}\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}\int_{1/f_2}^{1/f_1}\left(\frac{1}{f_1}-x\right)^{\alpha-1}\prod\left(\frac{1}{x}, \frac{2g_1g_2}{g_1+g_2}\right)\mathrm{d}x \leq \frac{\alpha\beta}{4}\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}\left(\frac{g_1g_2}{g_2-g_1}\right)^{\beta}$$
$$\times\left[\int_{1/f_2}^{1/f_1}\int_{1/g_2}^{1/g_1}\left(\frac{1}{f_1}-x\right)^{\alpha-1}\left(y-\frac{1}{g_2}\right)^{\beta-1}\prod\left(\frac{1}{x}, \frac{1}{y}\right)\mathrm{d}y\mathrm{d}x\right.$$
$$\left.+\int_{1/f_2}^{1/f_1}\int_{1/g_2}^{1/g_1}\left(\frac{1}{f_1}-x\right)^{\alpha-1}\left(\frac{1}{g_1}-y\right)^{\beta-1}\prod\left(\frac{1}{x}, \frac{1}{y}\right)\mathrm{d}y\mathrm{d}x\right]$$
$$\leq \frac{\alpha\beta}{4}\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}\left[\int_{1/f_2}^{1/f_1}\left(\frac{1}{f_1}-x\right)^{\alpha-1}\prod\left(\frac{1}{x}, g_1\right)\mathrm{d}x + \int_{1/f_2}^{1/f_1}\left(\frac{1}{f_1}-x\right)^{\alpha-1}\prod\left(\frac{1}{x}, g_2\right)\mathrm{d}x\right]. \tag{16}$$

Again by similar arguments for $\prod_{\frac{1}{y}} : [f_1, f_2] \to \mathbb{R}$, $\prod_{\frac{1}{y}}(x) = \prod(x, \frac{1}{y})$, we get

$$\frac{\beta}{2}\left(\frac{g_1g_2}{g_2-g_1}\right)^{\beta}\int_{1/g_2}^{1/g_1}\left(y-\frac{1}{g_2}\right)^{\beta-1}\prod\left(\frac{2f_1f_2}{f_1+f_2}, \frac{1}{y}\right)\mathrm{d}y$$
$$\leq \frac{\alpha\beta}{4}\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}\left(\frac{g_1g_2}{g_2-g_1}\right)^{\beta}\left[\int_{1/f_2}^{1/f_1}\int_{1/g_2}^{1/g_1}\left(u-\frac{1}{f_2}\right)^{\alpha-1}\left(y-\frac{1}{g_2}\right)^{\beta-1}\prod\left(\frac{1}{x}, \frac{1}{y}\right)\mathrm{d}y\mathrm{d}x\right.$$
$$\left.+\int_{1/f_2}^{1/f_1}\int_{1/g_2}^{1/g_1}\left(\frac{1}{f_1}-x\right)^{\alpha-1}\left(y-\frac{1}{g_2}\right)^{\beta-1}\prod\left(\frac{1}{x}, \frac{1}{y}\right)\mathrm{d}y\mathrm{d}x\right]$$

$$\leq \frac{\alpha\beta}{4}\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}\left[\int_{1/g_2}^{1/g_1}\left(y-\frac{1}{g_2}\right)^{\alpha-1}\prod\left(f_1,\frac{1}{y}\right)\mathrm{d}y+\int_{1/g_2}^{1/g_1}\left(y-\frac{1}{g_2}\right)^{\beta-1}\prod\left(f_2,\frac{1}{y}\right)\mathrm{d}y\right], \tag{17}$$

and

$$\begin{aligned}
&\frac{\beta}{2}\left(\frac{g_1g_2}{g_2-g_1}\right)^{\beta}\int_{1/g_2}^{1/g_1}\left(\frac{1}{g_1}-y\right)^{\beta-1}\prod\left(\frac{2f_1f_2}{f_1+f_2},\frac{1}{y}\right)\mathrm{d}y\\
&\leq \frac{\alpha\beta}{4}\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}\left(\frac{g_1g_2}{g_2-g_1}\right)^{\beta}\left[\int_{1/f_2}^{1/f_1}\int_{1/g_2}^{1/g_1}\left(x-\frac{1}{f_2}\right)^{\alpha-1}\left(\frac{1}{g_1}-y\right)^{\beta-1}\prod\left(\frac{1}{x},\frac{1}{y}\right)\mathrm{d}y\mathrm{d}x\right.\\
&\quad\left.+\int_{1/f_2}^{1/f_1}\int_{1/g_2}^{1/g_1}\left(\frac{1}{f_1}-x\right)^{\alpha-1}\left(\frac{1}{g_1}-y\right)^{\beta-1}\prod\left(\frac{1}{x},\frac{1}{y}\right)\mathrm{d}y\mathrm{d}x\right]\\
&\leq \frac{\alpha\beta}{4}\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}\left[\int_{1/g_2}^{1/g_1}\left(\frac{1}{g_1}-y\right)^{\alpha-1}\prod\left(f_1,\frac{1}{y}\right)\mathrm{d}y+\int_{1/g_2}^{1/g_1}\left(\frac{1}{g_1}-y\right)^{\beta-1}\prod\left(f_2,\frac{1}{y}\right)\mathrm{d}y\right].
\end{aligned} \tag{18}$$

By adding inequalities (15)–(18), we have

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{4}\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}\left[J^{\alpha}_{1/f_2+}(\prod\circ\Omega_1)\left(\frac{1}{f_1},\frac{2g_1g_2}{g_1+g_2}\right)+J^{\alpha}_{1/c_1-}(\prod\circ\Omega_1)\left(\frac{1}{f_2},\frac{2g_1g_2}{g_1+g_2}\right)\right]\\
&\quad+\frac{\Gamma(\beta+1)}{4}\left(\frac{g_1g_2}{g_2-g_1}\right)^{\beta}\left[J^{\beta}_{1/g_2+}(\prod\circ\Omega_2)\left(\frac{2f_1f_2}{f_1+f_2},\frac{1}{g_1}\right)+J^{\beta}_{1/g_1-}(\prod\circ\Omega_2)\left(\frac{2f_1f_2}{f_1+f_2},\frac{1}{g_2}\right)\right]\\
&\leq\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{2}\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}\left(\frac{g_1g_2}{g_2-g_1}\right)^{\beta}\times\left[J^{\alpha,\beta}_{f_1+,g_1+}(\prod\circ\Omega)\left(\frac{1}{f_2},\frac{1}{g_1}\right)+J^{\alpha,\beta}_{f_1+,g_2-}(\prod\circ\Omega)\left(\frac{1}{f_2},\frac{1}{g_1}\right)\right.\\
&\quad\left.+J^{\alpha,\beta}_{f_2-,g_1+}(\prod\circ\Omega)\left(\frac{1}{f_1},\frac{1}{g_2}\right)+J^{\alpha,\beta}_{f_2-,g_2-}(\prod\circ\Omega)\left(\frac{1}{f_1},\frac{1}{g_1}\right)\right]\\
&\leq\frac{\Gamma(\alpha+1)}{4}\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}\left[J^{\alpha}_{1/f_2+}(\prod\circ\Omega_1)\left(\frac{1}{f_1},g_2\right)+J^{\alpha}_{1/f_2+}(\prod\circ\Omega_1)\left(\frac{1}{f_1},g_1\right)\right.\\
&\quad\left.+J^{\alpha}_{1/f_1-}(\prod\circ\Omega_1)\left(\frac{1}{f_2},g_1\right)+J^{\alpha}_{1/f_1-}(\prod\circ\Omega_1)\left(\frac{1}{f_2},g_1\right)\right]\\
&\quad+\frac{\Gamma(\beta+1)}{4}\left(\frac{g_1g_2}{g_2-g_1}\right)^{\alpha}\left[J^{\beta}_{1/g_1-}(\prod\circ\Omega_2)\left(f_1,\frac{1}{g_2}\right)+J^{\beta}_{1/g_1-}(\prod\circ\Omega_2)\left(f_2,\frac{1}{g_2}\right)\right.\\
&\quad\left.+J^{\alpha}_{1/g_2+}(\prod\circ\Omega_2)\left(f_1,\frac{1}{g_1}\right)+J^{\alpha}_{1/g_2+}(\prod\circ\Omega_2)\left(f_2,\frac{1}{g_1}\right)\right].
\end{aligned} \tag{19}$$

This completes the second and third inequality of (12). Now again using (4), we have

$$\begin{aligned}
\prod\left(\frac{2f_1f_2}{f_1+f_2},\frac{2g_1g_2}{g_1+g_2}\right)\leq\frac{\alpha}{2}\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}&\left[\int_{1/f_2}^{1/f-1}\left(\frac{1}{f_1}-x\right)^{\alpha-1}\prod\left(\frac{1}{x},\frac{2g_1g_2}{g_1+g_2}\right)\mathrm{d}x\right.\\
&\left.+\int_{1/f_2}^{1/f_1}\left(x-\frac{1}{f_2}\right)^{\beta-1}\prod\left(\frac{1}{x},\frac{2g_1g_2}{g_1+g_2}\right)\mathrm{d}x\right],
\end{aligned} \tag{20}$$

$$\begin{aligned}
\prod\left(\frac{2f_1f_2}{f_1+f_2},\frac{2g_1g_2}{g_1+g_2}\right)\leq\frac{\beta}{2}\left(\frac{g_1g_2}{g_2-g_1}\right)^{\beta}&\left[\int_{1/g_2}^{1/g_1}\left(\frac{1}{g_1}-y\right)^{\beta-1}\prod\left(\frac{2f_1f_2}{f_1+f_2},\frac{1}{y}\right)\mathrm{d}y\right.\\
&\left.+\int_{1/g_2}^{1/g_1}\left(y-\frac{1}{g_2}\right)^{\beta-1}\prod\left(\frac{2f_1f_2}{f_1+f_2},\frac{1}{y}\right)\mathrm{d}y\right].
\end{aligned} \tag{21}$$

Adding (20) and (21), we get

$$\prod\left(\frac{2f_1f_2}{f_1+f_2},\frac{2g_1g_2}{g_1+g_2}\right)$$
$$\leq \frac{\Gamma(\alpha+1)}{4}\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}\left[J^{\alpha}_{1/f_2+}(\prod\circ\Omega_1)\left(\frac{1}{f_1},\frac{2g_1g_2}{g_1+g_2}\right)+J^{\alpha}_{1/f_1-}(\prod\circ\Omega_1)\left(\frac{1}{f_2},\frac{2g_1g_2}{g_1+g_2}\right)\right]$$
$$+\frac{\Gamma(\beta+1)}{4}\left(\frac{g_1g_2}{g_2-g_1}\right)^{\beta}\times\left[J^{\beta}_{1/g_2+}(\prod\circ\Omega_2)\left(\frac{2f_1f_2}{f_1+f_2},\frac{1}{g_1}\right)+J^{\beta}_{1/g_1-}(\prod\circ\Omega_2)\left(\frac{2f_1f_2}{f_1+f_2},\frac{1}{g_2}\right)\right]. \quad (22)$$

This completes the first inequality of (12). For the last inequality by using (4), we have

$$\frac{\alpha}{2}\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}\left[\int_{1/f_2}^{1/f_1}\left(\frac{1}{f_1}-x\right)^{\alpha-1}\prod\left(\frac{1}{x},g_1\right)\mathrm{d}x+\int_{1/f_2}^{1/f_1}\left(x-\frac{1}{f_2}\right)^{\beta-1}\prod\left(\frac{1}{x},g_1\right)\mathrm{d}x\right]$$
$$\leq\frac{\prod(f_1,g_1)+\prod(f_2,g_1)}{2},$$
$$\frac{\alpha}{2}\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}\left[\int_{1/f_2}^{1/f_1}\left(\frac{1}{f_1}-x\right)^{\alpha-1}\prod\left(\frac{1}{x},g_2\right)\mathrm{d}x+\int_{1/f_2}^{1/f_1}\left(x-\frac{1}{f_2}\right)^{\beta-1}\prod\left(\frac{1}{x},g_2\right)\mathrm{d}x\right]$$
$$\leq\frac{\prod(f_1,g_2)+\prod(f_2,g_2)}{2},$$
$$\frac{\beta}{2}\left(\frac{g_1g_2}{g_2-g_1}\right)^{\beta}\left[\int_{1/g_2}^{1/g_1}\left(\frac{1}{g_1}-y\right)^{\beta-1}\prod\left(f_1,\frac{1}{y}\right)\mathrm{d}y+\int_{1/g_2}^{1/g_1}\left(y-\frac{1}{g_2}\right)^{\beta-1}\prod\left(f_1,\frac{1}{y}\right)\mathrm{d}y\right]$$
$$\leq\frac{\prod(f_1,g_1)+\prod(f_1,g_2)}{2},$$
$$\frac{\beta}{2}\left(\frac{g_1g_2}{g_2-g_1}\right)^{\beta}\left[\int_{1/g_2}^{1/g_1}\left(\frac{1}{g_1}-y\right)^{\beta-1}\prod\left(f_2,\frac{1}{y}\right)\mathrm{d}y+\int_{1/g_2}^{1/g_1}\left(y-\frac{1}{g_2}\right)^{\beta-1}\prod\left(f_2,\frac{1}{y}\right)\mathrm{d}y\right]$$
$$\leq\frac{\prod(f_2,g_1)+\prod(f_2,g_2)}{2}.$$

Thus by adding all above inequalities, we get the last inequality of (12). Hence the proof is completed. □

Lemma 1. *Let* $\prod:\Delta=[f_1,f_2]\times[g_1,g_2]\subseteq(0,\infty)\times(0,\infty)\to\mathbb{R}$ *be a partial differentiable mapping on* Δ *with* $0<f_1<f_2$ *and* $0<g_1<g_2$*. If* $\partial^2\prod/\partial t_1\partial t_2\in L_1(\Delta)$*, then following holds:*

$$\frac{\prod(f_1,g_1)+\prod(f_1,g_2)+\prod(f_2,g_1)+\prod(f_2,g_2)}{4}+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4}\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}\left(\frac{g_1g_2}{g_2-g_1}\right)^{\beta}$$
$$\times\left[J^{\alpha,\beta}_{1/f_2+,1/g_1+}(\prod\circ\Omega)\left(\frac{1}{f_1},\frac{1}{g_1}\right)+J^{\alpha,\beta}_{1/f_1-,1/g_2+}(\prod\circ\Omega)\left(\frac{1}{f_2},\frac{1}{g_1}\right)\right.$$
$$\left.+J^{\alpha,\beta}_{1/f_2+,1/g_1-}(\prod\circ\Omega)\left(\frac{1}{f_1},\frac{1}{g_2}\right)+J^{\alpha,\beta}_{1/f_1-,1/g_1-}(\prod\circ\Omega)\left(\frac{1}{f_2},\frac{1}{g_2}\right)\right]-\Xi$$
$$=\frac{f_1f_2g_1g_2(f_2-f_1)(g_2-g_1)}{4}\left[\int_0^1\int_0^1\frac{r_1^{\alpha}r_2^{\beta}}{A_{t_1}^2B_{t_2}^2}\frac{\partial^2\prod}{\partial t_1\partial t_2}\left(\frac{f_1f_2}{A_{t_1}},\frac{g_1g_2}{B_{t_2}}\right)dt_2dt_1\right.$$
$$-\int_0^1\int_0^1\frac{(1-t_1)^{\alpha}t_2^{\beta}}{A_{t_1}^2B_{t_2}^2}\frac{\partial^2\prod}{\partial t_1\partial t_2}\left(\frac{f_1f_2}{A_{t_1}},\frac{g_1g_2}{B_{t_2}}\right)dt_2dt_1-\int_0^1\int_0^1\frac{t_1^{\alpha}(1-t_2)^{\beta}}{A_{t_1}^2B_{t_2}^2}\frac{\partial^2\prod}{\partial t_1\partial t_2}\left(\frac{f_1f_2}{A_{t_1}},\frac{g_1g_2}{B_{t_2}}\right)dt_2dt_1$$
$$\left.+\int_0^1\int_0^1\frac{(1-t_1)^{\alpha}(1-t_2)^{\beta}}{A_{t_1}^2B_{t_2}^2}\frac{\partial^2\prod}{\partial t_1\partial t_2}\left(\frac{f_1f_2}{A_{t_1}},\frac{g_1g_2}{B_{t_2}}\right)dt_2dt_1\right], \quad (23)$$

where

$$\Xi = \frac{\Gamma(\alpha+1)}{4}\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}\Bigg[J^{\alpha}_{1/f_2+}(\prod\circ\Omega_1)\left(\frac{1}{f_1},g_2\right)+J^{\alpha}_{1/f_1-}(\prod\circ\Omega_1)\left(\frac{1}{f_2},g_2\right)+J^{\alpha}_{1/f_2+}(\prod\circ\Omega_1)\left(\frac{1}{f_1},g_1\right)$$
$$+J^{\alpha}_{1/f_1-}(\prod\circ\Omega_1)\left(\frac{1}{f_2},g_1\right)\Bigg]+\frac{\Gamma(\beta+1)}{4}\left(\frac{g_1g_2}{g_2-g_1}\right)^{\beta}\Bigg[J^{\beta}_{1/g_2+}(\prod\circ\Omega_2)\left(f_2,\frac{1}{g_1}\right)$$
$$+J^{\beta}_{1/d_2+}(\prod\circ\Omega_2)\left(f_1,\frac{1}{g_1}\right)+J^{\beta}_{1/d_1-}(\prod\circ\Omega_2)\left(f_2,\frac{1}{g_2}\right)+J^{\beta}_{1/g_1-}(\prod\circ\Omega_2)\left(f_1,\frac{1}{g_2}\right)\Bigg], \quad (24)$$

and $A_{t_1}=t_1f_1+(1-t_1)f_2$, $B_{t_2}=t_2c+(1-t_2)d$. *Also,* $g(x,y)=(\frac{1}{x},\frac{1}{y})$, $g_1(x,y)=(\frac{1}{x},y)$, *and* $g_2(x,y)=(x,\frac{1}{y})$ *for all* $(x,y)\in\Delta$.

Proof. By integration by parts and using the change of variable $x=\frac{A_{t_1}}{f_1f_2}$ and $y=\frac{B_{t_2}}{g_1g_2}$, we find that

$$I_1=\int_0^1\int_0^1\frac{t_1^{\alpha}t_2^{\beta}}{A_{t_1}^2B_{t_2}^2}\frac{\partial^2\prod}{\partial t_1\partial t_2}\left(\frac{f_1f_2}{A_{t_1}},\frac{g_1g_2}{B_{t_2}}\right)dt_2dt_1$$
$$=\int_0^1\frac{t_2^{\beta}}{B_{t_2}^2}\left\{\frac{t_1^{\alpha}}{f_1f_2(f_2-f_1)}\frac{\partial\prod}{\partial t_2}\left(\frac{f_1f_2}{A_{t_1}},\frac{g_1g_2}{B_{t_2}}\right)\Bigg|_0^1-\frac{\alpha}{f_1f_2(f_2-f_1)}\int_0^1t_1^{\alpha-1}\frac{\partial\prod}{\partial t_2}\left(\frac{f_1f_2}{A_{t_1}},\frac{g_1g_2}{B_{t_2}}\right)dt_1\right\}dt_2$$
$$=\frac{1}{f_1f_2(f_2-f_1)}\int_0^1\frac{t_2^{\beta}}{B_{t_2}^2}\frac{\partial\prod}{\partial t_2}\left(f_2,\frac{g_1g_2}{B_{t_2}}\right)dt_2-\frac{\alpha}{f_1f_2(f_2-f_1)}\int_0^1t_1^{\alpha-1}\left\{\int_0^1\frac{t_2^{\beta}}{B_{t_2}^2}\frac{\partial\prod}{\partial t_2}\left(\frac{f_1f_2}{A_{t_1}},\frac{g_1g_2}{B_{t_2}}\right)dt_2\right\}dt_1$$
$$=\frac{1}{f_1f_2g_1g_2(f_2-f_1)(g_2-g_1)}\prod(f_2,g_2)-\frac{\beta}{f_1f_2g_1g_2(f_2-f_1)(g_2-g_1)}\int_0^1t_2^{\beta-1}\prod\left(f_2,\frac{g_1g_2}{B_{t_2}}\right)dt_2$$
$$-\frac{\alpha}{f_1f_2g_1g_2(f_2-f_1)(g_2-g_1)}\int_0^1t_1^{\alpha-1}\prod\left(\frac{f_1f_2}{A_{t_1}},d\right)dt_1$$
$$+\frac{\alpha\beta}{f_1f_2g_1g_2(f_2-f_1)(g_2-g_1)}\int_0^1t_1^{\alpha-1}t_2^{\beta-1}\prod\left(\frac{f_1f_2}{A_{r_1}},\frac{g_1g_2}{B_{t_2}}\right)dt_2$$
$$=\frac{1}{f_1f_2g_1g_2(f_2-f_1)(g_2-g_1)}\times\Bigg[\prod(f_2,g_2)-\Gamma(\beta+1)\left(\frac{g_1g_2}{g_2-g_1}\right)^{\beta}J^{\beta}_{1/g_2+}(\prod\circ\Omega_2)\left(f_2,\frac{1}{g_1}\right)$$
$$-\Gamma(\alpha+1)\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}J^{\alpha}_{1/f_2+}(\prod\circ\Omega_1)\left(\frac{1}{f_1},g_2\right)+\Gamma(\alpha+1)\Gamma(\beta+1)$$
$$\times\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}\left(\frac{g_1g_2}{g_2-g_1}\right)^{\beta}J^{\alpha,\beta}_{1/f_2+,1/g_2+}(\prod\circ\Omega)\left(\frac{1}{f_1},\frac{1}{g_1}\right)\Bigg]. \quad (25)$$

Similarly, we can have

$$I_2=\int_0^1\int_0^1\frac{(1-t_1)^{\alpha}t_2^{\beta}}{A_{t_1}^2B_{t_2}^2}\frac{\partial^2\prod}{\partial t_1\partial t_2}\left(\frac{f_1f_2}{A_{t_1}},\frac{g_1g_2}{B_{t_2}}\right)dt_2dt_1$$
$$=\frac{1}{f_1f_2g_1g_2(f_2-f_1)(g_2-g_1)}\Bigg[-\prod(f_1,g_1)+\Gamma(\beta+1)\left(\frac{g_1g_2}{g_2-g_1}\right)^{\beta}J^{\beta}_{1/g_2+}(\prod\circ\Omega_2)\left(f_1,\frac{1}{g_1}\right)$$
$$+\Gamma(\alpha+1)\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}J^{\alpha}_{1/f_1-}(\prod\circ\Omega_1)\left(\frac{1}{f_2},g_2\right)-\Gamma(\alpha+1)\Gamma(\beta+1)$$
$$\times\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}\left(\frac{g_1g_2}{g_2-g_1}\right)^{\beta}J^{\alpha,\beta}_{1/f_2+,1/g_2+}(\prod\circ\Omega)\left(\frac{1}{f_1},\frac{1}{g_1}\right)\Bigg]. \quad (26)$$

$$I_3=\int_0^1\int_0^1\frac{t_1^{\alpha}(1-t_2)^{\beta}}{A_{t_1}^2B_{t_2}^2}\frac{\partial^2\prod}{\partial t_1\partial t_2}\left(\frac{f_1f_2}{A_{t_1}},\frac{g_1g_2}{B_{t_2}}\right)dt_2dt_1$$

$$
\begin{aligned}
&= \frac{1}{f_1f_2g_1g_2(f_2-f_1)(g_2-g_1)}\left[-\prod(f_2,g_1)+\Gamma(\beta+1)\left(\frac{g_1g_2}{g_2-g_1}\right)^{\beta} J^{\beta}_{1/g_1-}\left(\prod\circ\Omega_2\right)\left(f_2,\frac{1}{g_2}\right)\right.\\
&+\Gamma(\alpha+1)\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha} J^{\alpha}_{1/f_2+}\left(\prod\circ\Omega_1\right)\left(\frac{1}{f_1},g_1\right)-\Gamma(\alpha+1)\Gamma(\beta+1)\\
&\left.\times\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}\left(\frac{g_1g_2}{g_2-g_1}\right)^{\beta} J^{\alpha,\beta}_{1/f_2+,1/g_1-}\left(\prod\circ\Omega\right)\left(\frac{1}{f_1},\frac{1}{g_2}\right)\right].
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
I_4 &= \int_0^1\int_0^1 \frac{(1-t_1)^{\alpha}(1-t_2)^{\beta}}{A^2_{t_1}B^2_{t_2}}\frac{\partial^2\prod}{\partial t_1\partial t_2}\left(\frac{f_1f_2}{A_{t_1}},\frac{g_1g_2}{B_{t_2}}\right)dt_2dt_1\\
&= \frac{1}{f_1f_2g_1g_2(f_2-f_1)(g_2-g_1)}\left[\prod(f_1,g_2)-\Gamma(\beta+1)\left(\frac{g_1g_2}{g_2-g_1}\right)^{\beta} J^{\beta}_{1/g_1-}\left(\prod\circ\Omega_2\right)\left(f_1,\frac{1}{g_2}\right)\right.\\
&-\Gamma(\alpha+1)\left(\frac{f_1f_2}{f_1-f_1}\right)^{\alpha} J^{\alpha}_{1/f_1-}\left(\prod\circ\Omega_1\right)\left(\frac{1}{f_2},g_1\right)+\Gamma(\alpha+1)\Gamma(\beta+1)\\
&\left.\times\left(\frac{f_1f_2}{f_1-f_1}\right)^{\alpha}\left(\frac{g_1g_2}{g_2-g_1}\right)^{\beta} J^{\alpha,\beta}_{1/f_1-,1/g_1-}\left(\prod\circ\Omega\right)\left(\frac{1}{f_2},\frac{1}{g_2}\right)\right].
\end{aligned}
\tag{28}
$$

Thus from equalities (25)–(28), we have

$$
\begin{aligned}
I_1-I_2-I_3+I_4 &= \frac{\Pi(f_2,g_2)+\Pi(f_1,g_1)+\Pi(f_2,g_1)+\Pi(f_1,g_2)}{f_1f_2g_1g_2(f_2-f_1)(g_2-g_1)}-\frac{\Gamma(\beta+1)}{f_1f_2g_1g_2(f_2-f_1)(g_2-g_1)}\left(\frac{g_1g_2}{g_2-g_1}\right)^{\beta}\\
&\times\left[J^{\beta}_{1/g_2+}\left(\prod\circ\Omega_2\right)\left(f_2,\frac{1}{g_1}\right)+J^{\beta}_{1/g_2+}\left(\prod\circ\Omega_2\right)\left(f_1,\frac{1}{g_1}\right)+J^{\beta}_{1/g_1-}\left(\prod\circ\Omega_2\right)\left(f_2,\frac{1}{g_2}\right)+J^{\beta}_{1/g_1-}\left(\prod\circ\Omega_2\right)\right.\\
&\left.\times\left(f_1,\frac{1}{g_2}\right)\right]-\frac{\Gamma(\alpha+1)}{f_1f_2g_1g_2(f_2-f_1)(g_2-g_1)}\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}\left[J^{\alpha}_{1/f_2+}\left(\prod\circ\Omega_1\right)\left(\frac{1}{f_1},g_2\right)+J^{\alpha}_{1/f_1-}\left(\prod\circ\Omega_1\right)\right.\\
&\left.\times\left(\frac{1}{f_2},g_2\right)+J^{\alpha}_{1/f_2+}\left(\prod\circ\Omega_1\right)\left(\frac{1}{f_1},g_1\right)+J^{\alpha}_{1/f_1-}\left(\prod\circ\Omega_1\right)\left(\frac{1}{f_2},g_1\right)\right]+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{f_1f_2g_1g_2(f_2-f_1)(g_2-g_1)}\\
&\times\left[J^{\alpha,\beta}_{1/f_2+,1/g_2+}\left(\prod\circ\Omega\right)\left(\frac{1}{f_1},\frac{1}{g_1}\right)+J^{\alpha,\beta}_{1/f_1-,1/g_2+}\left(\prod\circ\Omega\right)\left(\frac{1}{f_2},\frac{1}{g_1}\right)\right.\\
&\left.+J^{\alpha,\beta}_{1/f_2+,1/g_1-}\left(\prod\circ\Omega\right)\left(\frac{1}{f_1},\frac{1}{g_2}\right)+J^{\alpha,\beta}_{1/f_1-,1/g_1-}\left(\prod\circ\Omega\right)\left(\frac{1}{f_2},\frac{1}{g_2}\right)\right].
\end{aligned}
\tag{29}
$$

Multiplying both sides of equality (29) by $\frac{f_1f_2g_1g_2(f_2-f_1)(g_2-g_1)}{4}$, we get the desired equality (23). □

Theorem 13. *Let $\prod:\Delta=[f_1,f_2]\times[g_1,g_2]\subseteq(0,\infty)\times(0,\infty)\to\mathbb{R}$ be a partial differentiable mapping on Δ with $0<f_1<f_2$ and $0<g_1<g_2$. If $\left|\partial^2\prod/\partial t_1\partial t_2\right|$ is a harmonically convex on the co-ordinates on Δ, then following holds:*

$$
\begin{aligned}
&\left|\frac{\Pi(f_1,g_1)+\Pi(f_1,g_2)+\Pi(f_2,g_1)+\Pi(f_2,g_2)}{4}+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4}\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}\left(\frac{g_1g_2}{g_2-g_1}\right)^{\beta}\right.\\
&\quad\times\left[J^{\alpha,\beta}_{1/f_2+,1/g_1+}\left(\prod\circ\Omega\right)\left(\frac{1}{f_1},\frac{1}{g_1}\right)+J^{\alpha,\beta}_{1/f_1-,1/g_2+}\left(\prod\circ\Omega\right)\left(\frac{1}{f_2},\frac{1}{g_1}\right)\right.\\
&\quad\left.\left.+J^{\alpha,\beta}_{1/f_2+,1/g_1-}\left(\prod\circ\Omega\right)\left(\frac{1}{f_1},\frac{1}{g_2}\right)+J^{\alpha,\beta}_{1/f_1-,1/g_1-}\left(\prod\circ\Omega\right)\left(\frac{1}{f_2},\frac{1}{g_2}\right)\right]-\Xi\right|\\
&\le\frac{f_1g_1(f_2-f_1)(g_2-g_1)}{4f_2g_2(\alpha+1)(\beta+1)(\alpha+2)(\beta+2)}\left[\vartheta_1\left|\frac{\partial^2\Pi}{\partial t_1\partial t_2}(f_1,g_1)\right|+\vartheta_2\left|\frac{\partial^2\Pi}{\partial t_1\partial t_2}(f_1,g_2)\right|\right.\\
&\quad\left.+\vartheta_3\left|\frac{\partial^2\Pi}{\partial t_1\partial t_2}(f_2,g_1)\right|+\vartheta_4\left|\frac{\partial^2\Pi}{\partial t_1\partial t_2}(f_2,g_2)\right|\right],
\end{aligned}
\tag{30}
$$

where

$$\begin{aligned}\vartheta_1 =& (\alpha+1)(\beta+1)\ {}_2F_1\left(2,\alpha+2;\alpha+3;1-\frac{f_1}{f_2}\right)\ {}_2F_1\left(2,\beta+2;\beta+3;1-\frac{g_1}{g_2}\right)\\ &+(\beta+1)\ {}_2F_1\left(2,2;\alpha+3;1-\frac{f_1}{f_2}\right)\ {}_2F_1\left(2,\beta+2;\beta+3;1-\frac{g_1}{g_2}\right)+\ {}_2F_1\left(2,\alpha+2;\alpha+3;1-\frac{f_1}{f_2}\right)\\ &\times\ {}_2F_1\left(2,2;\beta+3;1-\frac{g_1}{g_2}\right)+\ {}_2F_1\left(2,2;\alpha+3;1-\frac{f_1}{f_2}\right)\ {}_2F_1\left(2,2;\beta+3;1-\frac{g_1}{g_2}\right), \end{aligned} \tag{31}$$

$$\begin{aligned}\vartheta_2 =& (\beta+1)\ {}_2F_1\left(2,\alpha+1;\alpha+3;1-\frac{f_1}{f_2}\right)\ {}_2F_1\left(2,\beta+2;\beta+3;1-\frac{g_1}{g_2}\right)\\ &+(\alpha+1)(\beta+1)\ {}_2F_1\left(2,1;\alpha+3;1-\frac{f_1}{f_2}\right)\ {}_2F_1\left(2,\beta+2;\beta+3;1-\frac{g_1}{g_2}\right)+\ {}_2F_1\left(2,\alpha+1;\alpha+3;1-\frac{f_1}{f_2}\right)\\ &\times\ {}_2F_1\left(2,2;\beta+3;1-\frac{g_1}{g_2}\right)+\ {}_2F_1\left(2,1;\alpha+3;1-\frac{f_1}{f_2}\right)\ {}_2F_1\left(2,2;\beta+3;1-\frac{g_1}{g_2}\right), \end{aligned} \tag{32}$$

$$\begin{aligned}\vartheta_3 =& (\alpha+1)\ {}_2F_1\left(2,\alpha+2;\alpha+3;1-\frac{f_1}{f_2}\right)\ {}_2F_1\left(2,\beta+1;\beta+3;1-\frac{g_1}{g_2}\right)\\ &+(\beta+1)\ {}_2F_1\left(2,2;\alpha+3;1-\frac{f_1}{f_2}\right)\ {}_2F_1\left(2,\beta+1;\beta+3;1-\frac{g_1}{g_2}\right)+(\beta+1)\ {}_2F_1\left(2,\alpha+2;\alpha+3;1-\frac{f_1}{f_2}\right)\\ &\times\ {}_2F_1\left(2,1;\beta+3;1-\frac{g_1}{g_2}\right)+\ {}_2F_1\left(2,2;\alpha+3;1-\frac{f_1}{f_2}\right)\ {}_2F_1\left(2,1;\beta+3;1-\frac{g_1}{g_2}\right), \end{aligned} \tag{33}$$

$$\begin{aligned}\vartheta_4 =&\ {}_2F_1\left(2,\alpha+1;\alpha+3;1-\frac{f_1}{f_2}\right)\ {}_2F_1\left(2,\beta+1;\beta+3;1-\frac{g_1}{g_2}\right)\\ &+(\alpha+1)\ {}_2F_1\left(2,1;\alpha+3;1-\frac{f_1}{f_2}\right)\ {}_2F_1\left(2,\beta+1;\beta+3;1-\frac{g_1}{g_2}\right)+(\beta+1)\ {}_2F_1\left(2,\alpha+1;\alpha+3;1-\frac{f_1}{f_2}\right)\\ &\times\ {}_2F_1\left(2,1;\beta+3;1-\frac{g_1}{g_2}\right)+(\alpha+1)(\beta+1)\ {}_2F_1\left(2,1;\alpha+3;1-\frac{f_1}{f_2}\right)\ {}_2F_1\left(2,1;\beta+3;1-\frac{g_1}{g_2}\right). \end{aligned} \tag{34}$$

Proof. Using Lemma 1, we have

$$\begin{aligned}&\frac{\prod(f_1,g_1)+\prod(f_1,g_2)+\prod(f_2,g_1)+\prod(f_2,g_2)}{4}+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4}\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}\left(\frac{g_1g_2}{g_2-g_1}\right)^{\beta}\\ &\times\left[J^{\alpha,\beta}_{1/f_2+,1/g_1+}(\prod\circ\Omega)\left(\frac{1}{f_1},\frac{1}{g_1}\right)+J^{\alpha,\beta}_{1/f_1-,1/g_2+}(\prod\circ\Omega)\left(\frac{1}{f_2},\frac{1}{g_1}\right)\right.\\ &\left.+J^{\alpha,\beta}_{1/f_2+,1/g_1-}(\prod\circ\Omega)\left(\frac{1}{f_1},\frac{1}{g_2}\right)+J^{\alpha,\beta}_{1/f_1-,1/g_1-}(\prod\circ\Omega)\left(\frac{1}{f_2},\frac{1}{g_2}\right)\right]-\Xi\\ &=\frac{f_1f_2g_1g_2(f_2-f_1)(g_2-g_1)}{4}\left[\int_0^1\int_0^1\frac{r_1^{\alpha}r_2^{\beta}}{A_{t_1}^2B_{t_2}^2}\frac{\partial^2\prod}{\partial t_1\partial t_2}\left(\frac{f_1f_2}{A_{t_1}},\frac{g_1g_2}{B_{t_2}}\right)\mathrm{d}t_2\mathrm{d}t_1\right.\\ &+\int_0^1\int_0^1\frac{(1-t_1)^{\alpha}t_2^{\beta}}{A_{t_1}^2B_{t_2}^2}\frac{\partial^2\prod}{\partial t_1\partial t_2}\left(\frac{f_1f_2}{A_{t_1}},\frac{g_1g_2}{B_{t_2}}\right)\mathrm{d}t_2\mathrm{d}t_1+\int_0^1\int_0^1\frac{t_1^{\alpha}(1-t_2)^{\beta}}{A_{t_1}^2B_{t_2}^2}\frac{\partial^2\prod}{\partial t_1\partial t_2}\left(\frac{f_1f_2}{A_{t_1}},\frac{g_1g_2}{B_{t_2}}\right)\mathrm{d}t_2\mathrm{d}t_1\\ &\left.+\int_0^1\int_0^1\frac{(1-t_1)^{\alpha}(1-t_2)^{\beta}}{A_{t_1}^2B_{t_2}^2}\frac{\partial^2\prod}{\partial t_1\partial t_2}\left(\frac{f_1f_2}{A_{t_1}},\frac{g_1g_2}{B_{t_2}}\right)\mathrm{d}t_2\mathrm{d}t_1\right].\end{aligned} \tag{35}$$

Now using co-ordinated harmonically convexity of $\left|\frac{\partial^2\prod}{\partial t_1\partial t_2}\right|$, we get

$$\left|\frac{\prod(f_1,g_1)+\prod(f_1,g_2)+\prod(f_2,g_1)+\prod(f_2,g_2)}{4}+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4}\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}\left(\frac{g_1g_2}{g_2-g_1}\right)^{\beta}\right.$$
$$\times\left[J^{\alpha,\beta}_{1/f_2+,1/g_1+}(\prod\circ\Omega)\left(\frac{1}{f_1},\frac{1}{g_1}\right)+J^{\alpha,\beta}_{1/f_1-,1/g_2+}(\prod\circ\Omega)\left(\frac{1}{f_2},\frac{1}{g_1}\right)\right.$$
$$\left.\left.+J^{\alpha,\beta}_{1/f_2+,1/g_1-}(\prod\circ\Omega)\left(\frac{1}{f_1},\frac{1}{g_2}\right)+J^{\alpha,\beta}_{1/f_1-,1/g_1-}(\prod\circ\Omega)\left(\frac{1}{f_2},\frac{1}{g_2}\right)\right]-\Xi\right|$$
$$\leq\frac{f_1f_2g_1g_2(f_2-f_1)(g_2-g_1)}{4}\left[\int_0^1\int_0^1\left\{\frac{t_1^{\alpha}t_2^{\beta}}{A_{t_1}^2B_{t_2}^2}+\frac{(1-t_1)^{\alpha}t_2^{\beta}}{A_{t_1}^2B_{t_2}^2}+\frac{t_1^{\alpha}(1-t_2)^{\beta}}{A_{t_1}^2B_{t_2}^2}\right.\right.$$
$$\left.+\frac{(1-t_1)^{\alpha}(1-t_2)^{\beta}}{A_{t_1}^2B_{t_2}^2}\right\}\left\{t_1t_2\left|\frac{\partial^2\prod}{\partial t_1\partial t_2}(f_1,g_1)\right|+(1-t_1)t_2\left|\frac{\partial^2\prod}{\partial t_1\partial t_2}(f_2,g_1)\right|\right.$$
$$\left.\left.+t_1(1-t_2)\left|\frac{\partial^2\prod}{\partial t_1\partial t_2}(f_1,g_2)\right|+(1-t_1)(1-t_2)\left|\frac{\partial^2\prod}{\partial t_1\partial t_2}(f_2,g_2)\right|\right\}dt_2dt_1\right]$$
$$=\frac{f_1f_2g_1g_2(f_2-f_1)(g_2-g_1)}{4}\left[\int_0^1\int_0^1 t_1t_2\left\{\frac{t_1^{\alpha}t_2^{\beta}}{A_{t_1}^2B_{t_2}^2}+\frac{(1-t_1)^{\alpha}t_2^{\beta}}{A_{t_1}^2B_{t_2}^2}+\frac{t_1^{\alpha}(1-t_2)^{\beta}}{A_{t_1}^2B_{t_2}^2}+\frac{(1-t_1)^{\alpha}(1-t_2)^{\beta}}{A_{t_1}^2B_{t_2}^2}\right\}\right.$$
$$\times\left|\frac{\partial^2\prod}{\partial t_1\partial t_2}(f_1,g_1)\right|dt_1dt_2+\int_0^1\int_0^1(1-t_1)t_2\left\{\frac{t_1^{\alpha}t_2^{\beta}}{A_{t_1}^2B_{t_2}^2}+\frac{(1-t_1)^{\alpha}t_2^{\beta}}{A_{t_1}^2B_{t_2}^2}+\frac{t_1^{\alpha}(1-t_2)^{\beta}}{A_{t_1}^2B_{t_2}^2}+\frac{(1-t_1)^{\alpha}(1-t_2)^{\beta}}{A_{t_1}^2B_{t_2}^2}\right\}$$
$$\times\left|\frac{\partial^2\prod}{\partial t_1\partial t_2}(f_2,g_1)\right|dt_1dt_2+\int_0^1\int_0^1 t_1(1-t_2)\left\{\frac{t_1^{\alpha}t_2^{\beta}}{A_{t_1}^2B_{t_2}^2}+\frac{(1-t_1)^{\alpha}t_2^{\beta}}{A_{t_1}^2B_{t_2}^2}+\frac{t_1^{\alpha}(1-t_2)^{\beta}}{A_{t_1}^2B_{t_2}^2}+\frac{(1-t_1)^{\alpha}(1-t_2)^{\beta}}{A_{t_1}^2B_{t_2}^2}\right\}$$
$$\times\left|\frac{\partial^2\prod}{\partial t_1\partial t_2}(f_1,g_2)\right|dt_1dt_2+\int_0^1\int_0^1(1-t_1)(1-t_2)$$
$$\left.\times\left\{\frac{t_1^{\alpha}t_2^{\beta}}{A_{t_1}^2B_{t_2}^2}+\frac{(1-t_1)^{\alpha}t_2^{\beta}}{A_{t_1}^2B_{t_2}^2}+\frac{t_1^{\alpha}(1-t_2)^{\beta}}{A_{t_1}^2B_{t_2}^2}+\frac{(1-t_1)^{\alpha}(1-t_2)^{\beta}}{A_{t_1}^2B_{t_2}^2}\right\}\left|\frac{\partial^2\prod}{\partial t_1\partial t_2}(f_2,g_2)\right|dt_1dt_2\right].\tag{36}$$

After calculating above integrations, we get the required result. □

Theorem 14. *Let* $\prod:\Delta=[f_1,f_2]\times[g_1,g_2]\subseteq(0,\infty)\times(0,\infty)\to\mathbb{R}$ *be a partial differentiable mapping on* Δ *with* $0<f_1<f_2$ *and* $0<g_1<g_2$*. If* $|\partial^2\prod/\partial t_1\partial t_2|^q$*,* $q>1$*, is a harmonically convex on the co-ordinates on* Δ*, then following holds:*

$$\left|\frac{\prod(f_1,g_1)+\prod(f_1,g_2)+\prod(f_2,g_1)+\prod(f_2,g_2)}{4}+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4}\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}\left(\frac{g_1g_2}{g_2-g_1}\right)^{\beta}\right.$$
$$\times\left[J^{\alpha,\beta}_{1/f_2+,1/g_1+}(\prod\circ\Omega)\left(\frac{1}{f_1},\frac{1}{g_1}\right)+J^{\alpha,\beta}_{1/f_1-,1/g_2+}(\prod\circ\Omega)\left(\frac{1}{f_2},\frac{1}{g_1}\right)\right.$$
$$\left.\left.+J^{\alpha,\beta}_{1/f_2+,1/g_1-}(\prod\circ\Omega)\left(\frac{1}{f_1},\frac{1}{g_2}\right)+J^{\alpha,\beta}_{1/f_1-,1/g_1-}(\prod\circ\Omega)\left(\frac{1}{f_2},\frac{1}{g_2}\right)\right]-\Xi\right|$$
$$\leq\frac{f_1g_1(f_2-f_1)(g_2-g_1)}{4f_2g_2[(p\alpha+1)(p\beta+1)]^{1/p}}\left[\psi_1^{1/p}+\psi_2^{1/p}+\psi_3^{1/p}+\psi_4^{1/p}\right]$$
$$\times\left(\frac{\left|\frac{\partial^2\prod}{\partial t_1\partial t_2}(f_1,g_1)\right|^q+\left|\frac{\partial^2\prod}{\partial t_1\partial t_2}(f_1,g_2)\right|^q+\left|\frac{\partial^2\prod}{\partial t_1\partial t_2}(f_2,g_1)\right|^q+\left|\frac{\partial^2\prod}{\partial t_1\partial t_2}(f_2,g_2)\right|^q}{4}\right)^{1/q},\tag{37}$$

where

$$\psi_1={}_2F_1\left(2p,p\alpha+1;p\alpha+2;1-\frac{f_1}{f_2}\right)\,{}_2F_1\left(2p,p\beta+1;p\beta+2;1-\frac{g_1}{g_2}\right),\tag{38}$$

$$\psi_2 = {}_2F_1\left(2p, 1; p\alpha+2; 1-\frac{f_1}{f_2}\right) {}_2F_1\left(2p, p\beta+1; p\beta+2; 1-\frac{g_1}{g_2}\right), \tag{39}$$

$$\psi_3 = {}_2F_1\left(2p, p\alpha+1; p\alpha+2; 1-\frac{f_1}{f_2}\right) {}_2F_1\left(2p, 1; p\beta+2; 1-\frac{g_1}{g_2}\right), \tag{40}$$

$$\psi_4 = {}_2F_1\left(2p, 1; p\alpha+2; 1-\frac{f_1}{f_2}\right) {}_2F_1\left(2p, 1; p\beta+2; 1-\frac{g_1}{g_2}\right). \tag{41}$$

Proof. Applying the Holder's inequality for double integrals in (35), we get

$$\begin{aligned}
&\Bigg|\frac{\Pi(f_1,g_1)+\Pi(f_1,g_2)+\Pi(f_2,g_1)+\Pi(f_2,g_2)}{4} + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4}\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}\left(\frac{g_1g_2}{g_2-g_1}\right)^{\beta} \\
&\quad\times\left[J^{\alpha,\beta}_{1/f_2+,1/g_1+}(\textstyle\prod\circ\Omega)\left(\frac{1}{f_1},\frac{1}{g_1}\right)+J^{\alpha,\beta}_{1/f_1-,1/g_2+}(\textstyle\prod\circ\Omega)\left(\frac{1}{f_2},\frac{1}{g_1}\right)\right. \\
&\quad\left.+J^{\alpha,\beta}_{1/f_2+,1/g_1-}(\textstyle\prod\circ\Omega)\left(\frac{1}{f_1},\frac{1}{g_2}\right)+J^{\alpha,\beta}_{1/f_1-,1/g_1-}(\textstyle\prod\circ\Omega)\left(\frac{1}{f_2},\frac{1}{g_2}\right)\right]-\Xi\Bigg| \\
&\le \frac{f_1f_2g_1g_2(f_2-f_1)(g_2-g_1)}{4}\left[\left(\int_0^1\int_0^1 \frac{t_1^{p\alpha}t_2^{p\beta}}{A_{t_1}^{2p}B_{t_2}^{2p}}dt_2dt_1\right)^{1/p}+\left(\int_0^1\int_0^1 \frac{(1-t_1)^{p\alpha}t_2^{p\beta}}{A_{t_1}^{2p}B_{t_2}^{2p}}dt_2dt_1\right)^{1/p}\right. \\
&\quad\left.+\left(\int_0^1\int_0^1 \frac{t_1^{p\alpha}(1-t_2)^{p\beta}}{A_{t_1}^{2p}B_{t_2}^{2p}}dt_2dt_1\right)^{1/p}+\left(\int_0^1\int_0^1 \frac{(1-t_1)^{p\alpha}(1-t_2)^{p\beta}}{A_{t_1}^{2p}B_{t_2}^{2p}}dt_2dt_1\right)^{1/p}\right] \\
&\quad\times\left(\int_0^1\int_0^1\left|\frac{\partial^2\Pi}{\partial t_1\partial t_2}\left(\frac{f_1f_2}{A_{t_1}},\frac{g_1g_2}{B_{t_2}}\right)\right|^q dt_1dt_2\right)^{1/q}.
\end{aligned} \tag{42}$$

Using co-ordinated harmonically convexity of $\left|\frac{\partial^2\Pi}{\partial t_1\partial t_2}\right|^q$, we get

$$\begin{aligned}
&\Bigg|\frac{\Pi(f_1,g_1)+\Pi(f_1,g_2)+\Pi(f_2,g_1)+\Pi(f_2,g_2)}{4} + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4}\left(\frac{f_1f_2}{f_2-f_1}\right)^{\alpha}\left(\frac{g_1g_2}{g_2-g_1}\right)^{\beta} \\
&\quad\times\left[J^{\alpha,\beta}_{1/f_2+,1/g_1+}(\textstyle\prod\circ\Omega)\left(\frac{1}{f_1},\frac{1}{g_1}\right)+J^{\alpha,\beta}_{1/f_1-,1/g_2+}(\textstyle\prod\circ\Omega)\left(\frac{1}{f_2},\frac{1}{g_1}\right)\right. \\
&\quad\left.+J^{\alpha,\beta}_{1/f_2+,1/g_1-}(\textstyle\prod\circ\Omega)\left(\frac{1}{f_1},\frac{1}{g_2}\right)+J^{\alpha,\beta}_{1/f_1-,1/g_1-}(\textstyle\prod\circ\Omega)\left(\frac{1}{f_2},\frac{1}{g_2}\right)\right]-\Xi\Bigg| \\
&\le \frac{f_1f_2g_1g_2(f_2-f_1)(g_2-g_1)}{4}\left[\left(\int_0^1\int_0^1 \frac{t_1^{p\alpha}t_2^{p\beta}}{A_{t_1}^{2p}B_{t_2}^{2p}}dt_2dt_1\right)^{1/p}+\left(\int_0^1\int_0^1 \frac{(1-t_1)^{p\alpha}t_2^{p\beta}}{A_{t_1}^{2p}B_{t_2}^{2p}}dt_2dt_1\right)^{1/p}\right. \\
&\quad\left.+\left(\int_0^1\int_0^1 \frac{t_1^{p\alpha}(1-t_2)^{p\beta}}{A_{t_1}^{2p}B_{t_2}^{2p}}dt_2dt_1\right)^{1/p}+\left(\int_0^1\int_0^1 \frac{(1-t_1)^{p\alpha}(1-t_2)^{p\beta}}{A_{t_1}^{2p}B_{t_2}^{2p}}dt_2dt_1\right)^{1/p}\right] \\
&\quad\times\left(\int_0^1\int_0^1\left\{t_1t_2\left|\frac{\partial^2\Pi}{\partial t_1\partial t_2}(f_1,f_2)\right|^q+(1-t_1)t_2\left|\frac{\partial^2\Pi}{\partial t_1\partial t_2}(f_2,g_1)\right|^q\right.\right. \\
&\quad\left.\left.+t_1(1-t_2)\left|\frac{\partial^2\Pi}{\partial t_1\partial t_2}(f_1,g_2)\right|^q+(1-t_1)(1-t_2)\left|\frac{\partial^2\Pi}{\partial t_1\partial t_2}(f_2,g_2)\right|^q\right\}dt_2dt_1\right)^{1/q} \\
&= \frac{f_1f_2g_1g_2(f_2-f_1)(g_2-g_1)}{4}\left[\left(\int_0^1\int_0^1 \frac{t_1^{p\alpha}t_2^{p\beta}}{A_{t_1}^{2p}B_{t_2}^{2p}}dt_2dt_1\right)^{1/p}+\left(\int_0^1\int_0^1 \frac{(1-t_1)^{p\alpha}t_2^{p\beta}}{A_{t_1}^{2p}B_{t_2}^{2p}}dt_2dt_1\right)^{1/p}\right.
\end{aligned}$$

$$+\left(\int_0^1\int_0^1 \frac{t_1^{p\alpha}(1-t_2)^{p\beta}}{A_{t_1}^{2p}B_{t_2}^{2p}}dt_2dt_1\right)^{1/p}+\left(\int_0^1\int_0^1 \frac{(1-t_1)^{p\alpha}(1-t_2)^{p\beta}}{A_{t_1}^{2p}B_{t_2}^{2p}}dt_2dt_1\right)^{1/p}\Bigg]$$
$$\times\left(\frac{\left|\frac{\partial^2\Pi}{\partial t_1\partial t_2}(f_1,g_1)\right|^q+\left|\frac{\partial^2\Pi}{\partial t_1\partial t_2}(f_1,g_2)\right|^q+\left|\frac{\partial^2\Pi}{\partial t_1\partial t_2}(f_2,g_1)\right|^q+\left|\frac{\partial^2\Pi}{\partial t_1\partial t_2}(f_2,g_2)\right|^q}{4}\right)^{1/q}. \tag{43}$$

By calculating all integrals, we get the required result (37). □

3. Conclusion

In Theorem 11 and 12, we have proved some new Hermite-Hadamard type inequalities for co-ordinated harmonically convex on a rectangle via Riemann-Liouville fractional integrals. In Lemma 1, we have proved a fractional integral identity and then with the help of this Lemma 1 we proved some fractional Hermite-Hadamard type inequalities on the co-ordinates.

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