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A Semi Analytic Approach to Coupled Boundary Value Problem

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Abstract

The present problem is considered as a coupled boundary value problem and is analyzed using a semi analytic method. A series method is used to obtain the solution and region of validity is extended by suitable techniques. In this case of series solution the results obtained are better than pure numerical findings up to moderately large Reynolds numbers. The variation of physical parameters is discussed in detail.

Keywords

1. Introduction

The flow between porous discs has been studied by several authors. As in the case of porous pipes and porous channels, the governing equation reduces to a set of nonlinear ordinary differential equations. This problem was first studied by Batchelor [1] who generalized the solution of Von Karman [2]. Further this problem was analyzed by Stewartson [3] who obtained perturbation solution for the small Reynolds number. Flow between a rotating and a stationary disc has been studied by Phan-Thein and Bush [4]. The problem of suction of viscous in incompressible fluid through a rotating porous disc onto a rotating co-axial disc was studied by C. Y. Wang [5]. Here we view this problem as a coupled nonlinear boundary value problem.

In the present paper we have used semianalytical numerical technique to understand the effect of suction represented by a pair of nonlinear differential equations. For simple geometrics the semianalytical numerical methods proposed here provide better results. Van Dyke [6] and his associates have successfully used these series methods in unveiling important features of various types of fluid flows. Bujurke and Pai [7] has discussed similar geometrical problem with injection at the discs.

Recently in the analysis of flow in pipe, Pai and Katagi [8] have used series analysis satisfactorily. The phys-

ical problem considered in this paper is of great importance in Fluid Dynamics. The present study is a possible extension work of Wang [5]. The problem is solved using a special type of series. The series so generated has limited utility due to presence of a singularity and its region of validity is extended by analytic continuation.

2. Mathematical Formulation

As shown in the **Figure 1** we denote the spacing between the discs by $z = \pm L$ and rotating with the same angular velocity Ω. Fluid is withdrawn from both discs with velocity W. We shall assume the gap with $2L$ is small compared to diameter of the discs. So that end effects can be neglected. The flow field is symmetric about the $z = 0$.

The governing equations are

$$uu_r + wu_z - \frac{v^2}{r} = -\frac{P_r}{\rho} + v\left(\nabla^2 u - \frac{u}{r^2}\right) \tag{1.1}$$

$$uv_r + wv_z + \frac{uv}{r} = v\left[\nabla^2 v - \frac{v}{r^2}\right] \tag{1.2}$$

$$uw_r + ww_z = -\frac{P_z}{\rho} + v\nabla^2 W \tag{1.3}$$

$$(ru)_r + rw_z = 0 \tag{1.4}$$

where $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$, and u, v, w are velocity components. The boundary conditions are at $z = \pm L$, $u = 0$, $v = r\Omega$, $w = \pm W$.

For similarity transformation we consider Wang [5]

$$\begin{aligned}
u &= rf'(\eta)\frac{W}{L}, \\
v &= rg(\eta)\frac{W}{L}, \\
w &= -2f(\eta)W, \\
p &= -\rho r^2 A\frac{W^2}{(2L^2)} + vP(\eta),
\end{aligned} \tag{1.5}$$

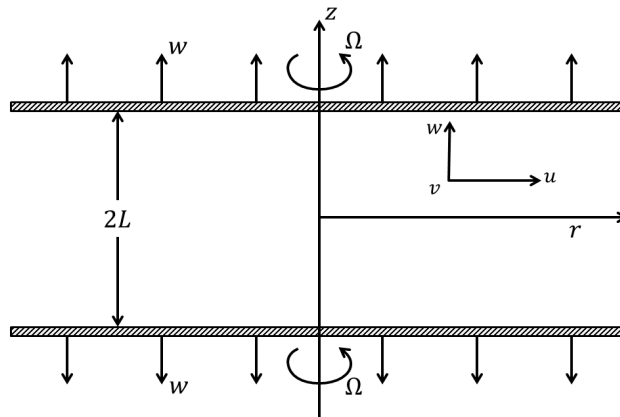where $\eta = \frac{z}{d}$ and A is constant. Now the equations take the form

Figure 1. The co-ordinate system.

$$\left(f'\right)^2 - 2ff'' - g^2 = A + \frac{1}{R}f'''$$

or

$$-2R\left(ff''' + gg'\right) = f^{iv} \qquad (1.6)$$

$$2R\left(f'g - fg'\right) = g'' \qquad (1.7)$$

$$P(\eta) = \left(-2f^2W^2 - 2vW\frac{f'}{L}\right) + B. \qquad (1.8)$$

Here $R = \dfrac{Wd}{v}$ is Reynolds number. The boundary conditions are

$$f'(1) = f(0) = f''(0) = 0, \quad f(1) = -\frac{1}{2}, \qquad (1.9)$$

$$g'(0) = 0, \quad g(1) = \frac{\Omega L}{W} = \beta. \qquad (1.10)$$

3. Method of Solution

We assume the solution of (1.6) and (1.7) in the forms

$$f(\eta) = f_0(\eta) + \sum_{n=1}^{\infty} R^n f_n(\eta) \qquad (3.1)$$

$$g(\eta) = g_0(\eta) + \sum_{n=1}^{\infty} R^n g_n(\eta). \qquad (3.2)$$

Substituting (3.1), (3.2) in (1.6) and (1.7) and comparing powers of R one can get

$$f_n^{iv} = -2\sum_{n=1}^{\infty} f_r f_{n-1-r}''' - g_r g_{n-1-r}' \qquad (3.3)$$

$$g_n'' = 2\sum_{n=1}^{\infty} \left(f_r' f_{n-1-r} - f_r g_{n-1-r}'\right) \quad n = 0,1,2\cdots. \qquad (3.4)$$

The relevant boundary conditions are

$$f'(1) = 0, \quad f_0(0) = f_0''(0) = 0, \quad f_0(1) = -\frac{1}{2},$$

$$g_0'(0) = 0, \quad g(1) = \beta,$$

$$f_n'(1) = 0, \quad f_n(0) = 0, \quad f_n''(0) = 0, \quad f_n(1) = 0,$$

$$g_n'(1) = 0 \,, \, g_n(1) = 0, \quad n = 1,2,3\cdots.$$

The solutions are

$$f_0 = \frac{1}{4}\left(\eta^3 - 3\eta\right), \quad g_0 = \beta,$$

$$f_1 = \frac{1}{1120}\left(-\eta^7 + 21\eta^5 - 39\eta^2 + 19\eta\right),$$

$$g_1 = \frac{\beta}{8}\left(\eta^4 - 6\eta^2 + 5\right),$$

$$f_2 = \frac{1}{560}\left\{\frac{-3}{440}\eta^{11} + \frac{1}{6}\eta^9 - \frac{177}{140}\eta^7 + \frac{17}{10}\eta^5 - \frac{443}{1848}\eta^3 - \frac{137}{385}\eta\right\} - \beta^2\left(\frac{1}{840}\eta^7 - \frac{1}{40}\eta^5 + \frac{13}{280}\eta^3 - \frac{19}{840}\eta\right),$$

$$g_2 = \beta \left(\frac{-3}{2240} \eta^8 + \frac{1}{80} \eta^6 - \frac{111}{3360} \eta^4 - \frac{253}{560} \eta^2 + \frac{3183}{6720} \right).$$

The slow convergence of series (3.1), (3.2) requires large number of terms for obtaining the approximate sum. For higher order approximations we proceed by the following scheme.

We propose a systematic scheme for this purpose, we consider f_n and g_n to be of the forms

$$f_n(\eta) = \sum_{k=2}^{4n+1} C_{n,k} \eta^k (1-\eta)^2 \tag{3.5}$$

$$g_n(\eta) = \sum_{n=1}^{4n} D_{n,k} \eta^k (1-\eta). \tag{3.6}$$

We substitute (3.5), (3.6) in (3.3), (3.4) respectively and equate various powers of η on both sides to get

$$C_{n(k)} = 2C_{n(k+1)} - C_{n(k+2)} + \frac{1}{(k+2)(K+1)(k)(k-1)} \times \left\{ \sum_{i=1}^{6} C_{(n-1)(k+2-i)} P_{i(k+2-i)} + \sum_{i=1}^{3} D_{(n-1)(k-i)} Q_{i(k-i)} \right. $$
$$\left. + \sum_{r=1}^{n-2} \left[\sum_{i=0}^{4} \sum_{j=1}^{nk+i} C_{r(\ell+1-nk-i)} C_{(m-1)(t-2)} P_{7+i}(t-2) + \sum_{r=0}^{2} \sum_{j=1}^{nk+i} D_{r(\ell-nk-i)(t-3)} D_{(m-1)(t-3)} Q_{(4+i)}(t-3) \right] \right\}, \tag{3.7}$$

$$D_{n(k)} = D_{n(k+1)} - \frac{1}{k(k+1)} \left[\sum_{i=1}^{5} D_{(n-1)(k+1-i)} T_i(k+1-i) + \sum_{i=1}^{4} C_{(n-1)(k+1-i)} T_i'(k+1-i) \right.$$
$$\left. + \sum_{r=1}^{n-2} \sum_{i=0}^{3} \sum_{j=1}^{nk+i} C_{(\ell+1-nk-i)} D_{(m-1)(t-3)} \Gamma_{6+i}(\ell+1-nk-i, t-3) \right], \tag{3.8}$$

where $m = n-4$, $l = 4r+j$, $t = 4m-j$, $nk = 4n-2-k$, P_i, Q_i, T_i's are functions of k and k_1.

For the radial velocity profile,

$$f'(\eta) = \frac{1}{4}(3\eta^2 - 3) + \sum_{n=1}^{\infty} R^n \sum_{k=2}^{4n+1} C_{n(k)} - (k\eta^{k-1} - 2(k+1)\eta^k + (k+2)\eta^{k+1}). \tag{3.9}$$
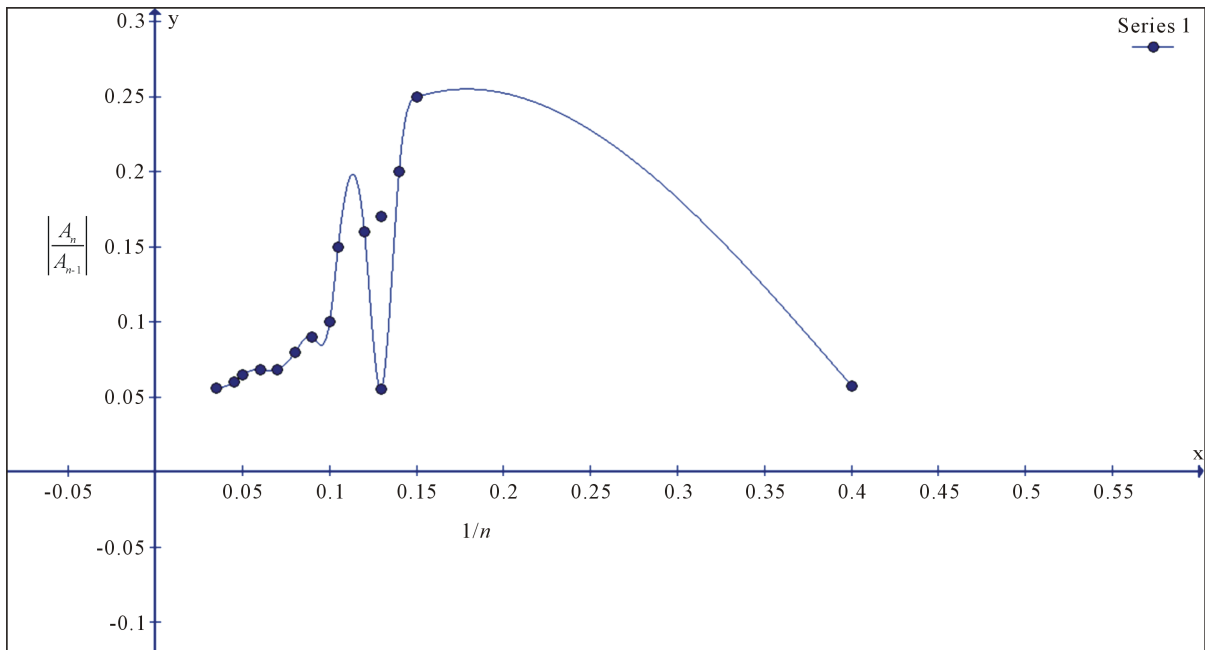

Figure 2. Domb sykes plot.

Table 1. Comparison of values for $f'(0)$.

R	β	$f'(0)$ (numerical)	Series method
40.812	3.00	---------	–0.34121
20.345	3.6648	–0.54398	–0.54398
0.13151	0.68501	–0.74150	–0.74150
–39.0691	0.98224	-0.6550	–0.65511
–60.000	0.98204	--------	–0.41511

Table 2. Comparison of values for $f'''(0)$.

R	β	$f'''(0)$ (numerical)	Series method
40.812	3.00	--------	0.0000
20.345	3.6648	0.0069	0.00688
8.5679	0.000	0.24870	0.24871
0.13151	0.68501	1.37862	1.37862
–4.39563	0.43440	2.24572	2.24571
–39.0691	0.98224	0.00156	0.00156
–60.000	0.98204	--------	0.00000

And we have

$$f'''(\eta) = \frac{3}{2} + \sum_{m=1}^{\infty} R^n \sum_{k=2}^{4n+1} C_{n(k)} \left(k(k-1)\ (k-2)\eta^{k-2} - 2(k+1)(k)(k-1)\eta^{k-1} + (k+2)(k+1)k\eta^k \right) \tag{3.10}$$

4. Conclusions

A new type of series is presented for studying the problems of flow between the two discs using recurrences (3.7) and (3.8). In this process we generate universal coefficients $C_{n(k)}$, $\eta = 1, \cdots, 30$, $k = 4n+1$.

These coefficients in turn give universal coefficients for $f_n(\eta)(n = 1, 2, \cdots, 30)$. The series (3.9) and (3.10) give $f'(0)$ and $f'''(0)$ respectively using Domb-sykes plot (**Figure 2**), and we locate the position of singularities for $\beta = 0$ at 17.89. The series (3.9) and (3.10) are summed using Pade-approximants. Earlier numerical results were from $R = 20$ to -39. We are able to go upto $R = 40$ to -60 using Pade-approximants. The results are in close agreement with numerical findings of Wang [5] which are shown in the **Table 1** and **Table 2**.

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