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A Cauchy Problem for Some Fractional *q*-Difference Equations with Nonlocal Conditions

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Abstract

In this paper, we discussed the problem of nonlocal value for nonlinear fractional *q*-difference equation. The classical tools of fixed point theorems such as Krasnoselskii's theorem and Banach's contraction principle are used. At the end of the manuscript, we have an example that illustrates the key findings.

Keywords

Cauchy Problem, Fractional *q*-Difference Equation, Nonlocal Conditions, Fixed Point, Krasnoselskii's Theorem

1. Introduction

Importance of fractional differential equations appears in many of the physical and engineering phenomena in the last two decades [1]-[3]. Problems with nonlocal conditions and related topics were studied in, for example [4], and the nonlocal Cauchy problem [5]. The attention of researchers subject of q-difference equations appeared in recent years [6] [7]. Initially, it was developed by Jackson [8] [9]. Noted recently the attention of many researchers is in the field of fractional q-calculus [10] [11]. Recently nonlocal fractional q-difference problems have aroused considerable attention [12] [13].

In this paper, we obtain the results of the existence and uniqueness of solutions for the Cauchy problem with nonlocal conditions for some fractional q-difference equations given by

$$\begin{cases} {}_C D_q^{\alpha} u(t) = f\left(t, u(t)\right), \ t \in I, \quad 0 < \alpha < 1, \\ u(0) + g(u) = u_0. \end{cases} \tag{1}$$

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Here, ${}_C D_q^\alpha$ is the Caputo fractional q-derivative of order α, $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ and $g:C([0,1],\mathbb{R})\to\mathbb{R}$ are given continuous functions. It is worth mentioning that the nonlocal condition $u(0)+g(u)=u_0$ which can be applied effectively in physics is better than the classical Cauchy problem condition $u(0)=u_0$, see [14].

Several authors have studied the semi-linear differential equations with nonlocal conditions in Banach space, [15] [16]. In [17], Dong *et al.* studied the existence and uniqueness of the solutions to the nonlocal problem for the fractional differential equation in Banach space. Motivated by these studied, we explore the Cauchy problem for nonlinear fractional q-difference equations according to the following hypotheses.

(H_1) $f:\mathbb{R}\times X\to X$ is jointly continuous.

(H_2) $\left\|f(t,u_1)-f(t,u_2)\right\|\le L\left\|u_1-u_2\right\|, \forall t\in\mathbb{R}; u_1,u_2\in X.$

(H_3) $g:C\to X$ is continuous and $\left\|g(u_1)-g(u_2)\right\|\le b\left\|u_1-u_2\right\|, \forall u_1,u_2\in C.$

(H_4) $\left\|f(t,u)\right\|\le\mu(t), \forall(t,u)\in I\times X,$ where $\mu\in L^1(I,\mathbb{R}^+)$.

The problem (1) is then devolved to the following formula

$$u(t)=u_0-g(u)+\frac{1}{\Gamma_q(\alpha)}\int_0^t(t-qs)^{(\alpha-1)}f(s,u(s))\mathrm{d}_q s, t\in[0,T]. \tag{2}$$

See reference [18] for more details.

2. Preliminaries on Fractional *q*-Calculus

Let $q\in(0,1)$ and define

$$[a]_q=\frac{q^a-1}{q-1}=q^{a-1}+\cdots+1, a\in\mathbb{R}.$$

The q-analogue of the Pochhammer symbol was presented as follows

$$(a;q)_0=1,\ (a;q)_n=\prod_{k=0}^{n-1}(1-aq^k), a\in\mathbb{R}, n\in\mathbb{N}\cup\{\infty\}.$$

In general, if $\alpha\in\mathbb{R}$ thereafter

$$(a;q)_\infty=\prod_{i=0}^{\infty}(1-aq^i),\quad (a;q)_\alpha=\frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}.$$

The q-gamma function is defined by

$$\Gamma_q(x)=(q;q)_{x-1}(1-q)^{1-x}, x\in\mathbb{R}\setminus\{0,-1,-2,\cdots\}, 0<q<1,$$

and satisfies $\Gamma_q(x+1)=[x]_q\Gamma_q(x)$.

The q-derivative of a function $f(x)$ is here defined by

$$D_q f(x)=\frac{\mathrm{d}_q f(x)}{\mathrm{d}_q x}=\frac{f(qx)-f(x)}{(q-1)x},$$

and

$$D_q^n f(x)=\begin{cases} f(x) & \text{if } n=0,\\ D_q D_q^{n-1}f(x) & \text{if } n\in\mathbb{N}.\end{cases}$$

The q-integral of a function *f* defined in the interval $[0,b]$ is provided by

$$\int_0^x f(t)\,\mathrm{d}_q t = x(1-q)\sum_{n=0}^{\infty} f(xq^n)q^n, \quad 0 \le |q| < 1, \quad x \in [0,b].$$

Now, it can be defined an operator I_q^n, as follows

$$\left(I_q^0 f\right)(x) = f(x) \text{ and } \left(I_q^n f\right)(x) = I_q\left(I_q^{n-1} f\right)(x), n \in \mathbb{N}.$$

We can point to the basic formula which will be used at a later time,

$${}_sD_q t^{\alpha}(s/t;q)_{\alpha} = -[\alpha]_q t^{\alpha-1}(qs/t;q)_{\alpha-1},$$

where ${}_sD_q$ denotes the q-derivative with respect to variable s.

See reference [7]-[10] for more details.

Definition 2.1. [19] Let $\alpha \ge 0$ and f be a function defined on $[0,1]$. The fractional q-integral of the Riemann-Liouville type is $\left(I_q^0 f\right)(x) = f(x)$ and

$$\left(I_q^{\alpha} f\right)(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1} f(t)\,\mathrm{d}_q t, \alpha \in \mathbb{R}^+, x \in [0,1].$$

Definition 2.3. [19] The fractional q-derivative of the Caputo type of order $\alpha > 0$ is defined by

$$\left({}_C D_q^{\alpha} f\right)(x) = \left(I_q^{[\alpha]-\alpha} D_q^{[\alpha]} f\right)(x).$$

where $[\alpha]$ is the smallest integer greater than or equal to α.

Theorem 2.1. [20] Let $x > 0$ and $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$. Then, the following equality holds

$$\left(I_q^{\alpha}\,{}_C D_q^{\alpha} f\right)(x) = f(x) - \sum_{k=0}^{[\alpha]-1} \frac{x^k}{\Gamma_q(k+1)}\left(D_q^k f\right)(0).$$

Theorem 2.2. [18] [19] **(Krasnoselskii)**

Let M be a closed convex non-empty subset of a Banach space $(X,\|.\|)$. Suppose that A and B maps M into X, such that the following hypotheses are fulfilled:

1) $Au_1 + Bu_2 \in M$ for all $u_1, u_2 \in M$;

2) A is continuous and AM is contained in a compact set;

3) B is a contraction mapping.

Then there exists $z \in M$ such that $z = Az + Bz$.

3. Main Results

Now, the obtained results are presented.

Theorem 3.1.

Let (H_1)- (H_3) hold, if $b < \frac{1}{2}$ and $L \le \frac{\Gamma_q(\alpha+1)}{2T^{\alpha}}$, the problem (1) has a unique solution.

Proof. Define $F: C \to C$ by

$$(Fu)(t) := u_0 - g(u) + \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} f(s,u(s))\,\mathrm{d}_q s.$$

Choose $r \ge 2\left(\|u_0\| + G + \frac{MT^{\alpha}}{\Gamma_q(\alpha+1)}\right)$ and let $M = \sup_{t \in I}|f(t,0)|$. So, we can prove that $FP \subset P$, where

$P := \{u \in C : \|u\| \le r, r > 0\}$. For it, let $u \in P$ and $G = \sup_{u \in C} |g(u)|$. Consequently, we find that

$$\begin{aligned}
\|Fu(t)\| &= \left\| u_0 - g(u) + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t (q\,s/t;q)_{\alpha-1} f(s,u(s)) \mathrm{d}_q s \right\| \\
&\le \|u_0\| + G + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t \left\| (q\,s/t;q)_{\alpha-1} f(s,u(s)) \right\| \mathrm{d}_q s \\
&\le \|u_0\| + G + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t (q\,s/t;q)_{\alpha-1} \left[\|f(s,u) - f(s,0)\| + \|f(s,0)\| \right] \mathrm{d}_q s \\
&\le \|u_0\| + G + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t (q\,s/t;q)_{\alpha-1} \left[L\|u\| + \sup_{s \in I} \|f(s,0)\| \right] \mathrm{d}_q s \\
&= \|u_0\| + G + \frac{(Lr+M)t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t (q\,s/t;q)_{\alpha-1} \mathrm{d}_q s \\
&= \|u_0\| + G + \frac{(Lr+M)}{\Gamma_q(\alpha+1)} t^{\alpha} \\
&\le \|u_0\| + G + \frac{MT^{\alpha}}{\Gamma_q(\alpha+1)} + \frac{rLT^{\alpha}}{\Gamma_q(\alpha+1)} \le r.
\end{aligned}$$

This shows that $\|Fu(t)\| \le r \; \forall u \in P$, therefore, $FP \subset P$.

Now, for $u_1, u_2 \in C$, we obtain

$$\begin{aligned}
&\|F(u_1)(t) - F(u_2)(t)\| \\
&\le \|g(u_1) - g(u_2)\| + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t (q\,s/t;q)_{\alpha-1} \|f(s,u_1(s)) - f(s,u_2(s))\| \mathrm{d}_q s \\
&\le b\|u_1 - u_2\| + \frac{L\|u_1 - u_2\| t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t (q\,s/t;q)_{\alpha-1} \mathrm{d}_q s \\
&\le \left[b + \frac{L}{\Gamma_q(\alpha+1)} T^{\alpha} \right] \|u_1 - u_2\| = \Omega_{b,L,T,\alpha,q} \|u_1 - u_2\|.
\end{aligned}$$

Thus

$$\|F(u_1)(t) - F(u_2)(t)\| \le K\|u_1 - u_2\|,$$

where $K = \Omega_{b,L,T,\alpha,q} = b + \frac{L}{\Gamma_q(\alpha+1)} T^{\alpha} < 1$.

Thus, by the Banach's contraction mapping principle, we find that the problem (1) has a unique solution.

Our next results are based on Krasnoselskii's fixed-point theorem.

Theorem 3.2.

Let (H_1), (H_2), (H_3) with $b < 1$ and (H_4) hold, then the problem (1) has at least one solution on *I*.

Proof. Take $r \ge \|u_0\| + G + \frac{MT^{\alpha}}{\Gamma_q(\alpha+1)}$, and consider $P := \{u \in C : \|u\| \le r, r > 0\}$.

Let *A* and *B* the two operators defined on *P* by

$$(Au)(t) := \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t (q\,s/t;q)_{\alpha-1} f(s,u(s)) \mathrm{d}_q s,$$

and

$$(Bu)(t) := u_0 - g(u),$$

respectively. Note that if $u_1, u_2 \in P$ then

$$\begin{aligned}\|Au_1 + Bu_2\| &= \left\| u_0 - g(u_2) + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t (qs/t;q)_{\alpha-1} f(s, u_1(s)) \mathrm{d}_q s \right\| \\ &\le \|u_0\| + \|g(u_2)\| + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t \left\| (qs/t;q)_{\alpha-1} f(s, u_1(s)) \right\| \mathrm{d}_q s \\ &\le \|u_0\| + G + \frac{\|\mu\|_{L^1} t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t (qs/t;q)_{\alpha-1} \mathrm{d}_q s \\ &\le \|u_0\| + G + \frac{\|\mu\|_{L^1} T^\alpha}{\Gamma_q(\alpha+1)} \le r.\end{aligned}$$

Thus $Au_1 + Bu_2 \in P$.

By (H_2), it is also clear that B is a contraction mapping.

Produced from Continuity of u, the operator $(Au)(t)$ is continuous in accordance with (H_1). Also we observe that

$$\|Au(t)\| \le \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t (qs/t;q)_{\alpha-1} \|f(s, u(s))\| \mathrm{d}_q s \le \frac{\|\mu\|_{L^1} T^\alpha}{\Gamma_q(\alpha+1)}.$$

Then A is uniformly bounded on P.

Now, let $t_1, t_2 \in I, t_2 \le t_1$ and $u \in P$. That's where f is bounded on the compact set $I \times P$. it means $\sup_{(t,u)\in I\times P} \|f(t,u)\| := c_0 < \infty.$ We will get

$$\begin{aligned}&\|A(u)(t_1) - A(u)(t_2)\| = \left\| \frac{t_1^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^{t_1} (qs/t_1;q)_{\alpha-1} f(s, u(s)) \mathrm{d}_q s - \frac{t_2^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^{t_2} (qs/t_2;q)_{\alpha-1} f(s, u(s)) d_q s \right\| \\ &= \left\| \frac{1}{\Gamma_q(\alpha)} \left[t_1^{\alpha-1} \int_0^{t_2} (qs/t_1;q)_{\alpha-1} f(s,u(s)) \mathrm{d}_q s + t_1^{\alpha-1} \int_{t_2}^{t_1} (qs/t_1;q)_{\alpha-1} f(s,u(s)) \mathrm{d}_q s - t_2^{\alpha-1} \int_0^{t_2} (qs/t_2;q)_{\alpha-1} f(s,u(s)) \mathrm{d}_q s \right] \right\| \\ &\le \frac{1}{\Gamma_q(\alpha)} \left[t_1^{\alpha-1} \int_{t_2}^{t_1} (qs/t_1;q)_{\alpha-1} \|f(s,u(s))\| \mathrm{d}_q s + \int_0^{t_2} \left(t_2^{\alpha-1} (qs/t_2;q)_{\alpha-1} - t_1^{\alpha-1} (qs/t_1;q)_{\alpha-1} \right) \|f(s,u(s))\| \mathrm{d}_q s \right] \\ &\le \frac{c_0}{\Gamma_q(\alpha)} \left[t_1^{\alpha-1} \int_{t_2}^{t_1} (qs/t_1;q)_{\alpha-1} \mathrm{d}_q s + \int_0^{t_2} \left(t_2^{\alpha-1} (qs/t_2;q)_{\alpha-1} - t_1^{\alpha-1} (qs/t_1;q)_{\alpha-1} \right) \right],\end{aligned}$$

which is autonomous of u and head for zero as $t_1 - t_2 \to 0.$ Consequently, A is equicontinuous. Thus, A is relatively compact on P. Therefore, according to the Arzela-Ascoli Theorem, A is compact on P. Thus, the problem (1) has at least one solution on I.

Example 4.1 Consider the following nonlocal problem

$$\begin{cases} {}_C D_q^\alpha u(t) = \dfrac{\mathrm{e}^{-t} |u(t)|}{(7+\mathrm{e}^t)(1+|u(t)|)}, \alpha \in (0,1), q \in (0,1), t \in I = [0,1], \\ u(0) + \sum_{i=1}^m a_i u(t_i) = 0, 0 < t_1 < t_2 < \cdots < t_m < 1, \end{cases} \tag{3}$$

where $a_i > 0, i = 1, 2, \cdots, m.$

Set

$$f(t,u) = \frac{\mathrm{e}^{-t} u}{(7+\mathrm{e}^t)(1+u)}, \quad (t,u) \in I \times [0,r],$$

and

$$g(u)=\sum_{i=1}^{m}a_i u(t_i).$$

Let $u_1,u_2\in X$ and $t\in I$. Then we have

$$\left|f(t,u_1)-f(t,u_2)\right|\le\frac{\mathrm{e}^{-t}}{9+\mathrm{e}^{t}}|u_1-u_2|\le\frac{1}{10}|u_1-u_2|,$$

and

$$\left|g(u_1)-g(u_2)\right|=\sum_{i=1}^{m}a_i\left|u_1(t_i)-u_2(t_i)\right|\le\sum_{i=1}^{m}a_i\max\left\{\left|u_1(t_i)-u_2(t_i)\right|\right\}.$$

It is obviously that our assumptions in Theorem 3.1 holds with $L=\frac{1}{10}$ and for appropriate values of $\alpha\in(0,1), q\in(0,1)$ with $T=1$ and $\sum a_i<\frac{1}{2}$. Indeed

$$L\le\frac{\Gamma_q(\alpha+1)}{2}\Leftrightarrow\Gamma_q(\alpha+1)\ge 2L=0.2. \tag{4}$$

Therefore the problem (3) has a unique solution on $[0,1]$ for values of α and q sufficient stipulation (4). For illustration

- If $\alpha=\frac{1}{8}$ and $q=\frac{1}{2}$ then $\Gamma_q(\alpha+1)=\Gamma_{0.5}\left(\frac{9}{8}\right)=0.957935$ and $\frac{\Gamma_q(\alpha+1)}{2}=\frac{0.957935}{2}=0.4789675>0.1.$
- If $\alpha=\frac{2}{5}$ and $q=\frac{1}{2}$ then $\Gamma_q(\alpha+1)=\Gamma_{0.5}\left(\frac{7}{5}\right)=0.920684$ and $\frac{\Gamma_q(\alpha+1)}{2}=\frac{0.920684}{2}=0.460342>0.1.$

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