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Almost Periodic Solution of a Discrete Multispecies Lotka-Volterra Competition Predator-prey System

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Abstract

In this paper, we consider an almost periodic discrete multispecies Lotka-Volterra competition predator-prey system. By the almost periodicity, sufficient conditions which guarantee the existence of a unique globally attractive almost periodic solution are obtained. An suitable example together with numerical simulation indicates the feasibility of the main results.

Keywords: Almost periodic solution; predator-prey system; discrete; permanence; global attractivity.
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1 Introduction

In 2006, Chen[1] had studied the following discrete $n+m$-species Lotka-Volerra competition predator-prey system

$$x_i(k+1) = x_i(k)\exp\left[b_i(k) - \sum_{l=1}^{n} a_{il}(k)x_l(k) - \sum_{l=1}^{m} c_{il}(k)y_l(k)\right],$$
$$y_j(k+1) = y_j(k)\exp\left[-r_j(k) + \sum_{l=1}^{n} d_{jl}(k)x_l(k) - \sum_{l=1}^{m} e_{jl}(k)y_l(k)\right], \quad (1.1)$$

where $i = 1, 2, \cdots, n$; $j = 1, 2, \cdots, m$; $x_i(k)$ is the density of prey species i at kth generation. $y_j(k)$ is the density of predator species j at kth generation. $a_{il}(k)$ and $e_{jl}(k)$ measures the intensity of intraspecific competition or interspecific action of prey species and predator species, respectively. $b_i(k)$ representing the intrinsic growth rate of the prey species $x_i(k)$; $r_j(k)$ representing the death rate of the predator species $y_j(k)$. Sufficient conditions which ensure the permanence and the global stability of systems (1.1) are obtained; for periodic case, sufficient conditions which ensure the existence of a globally stable positive periodic solution of the systems are obtained.

In real world phenomenon, the environment varies due to the factors such as seasonal effects of weather, food supplies, mating habits, harvesting. So it is usual to assume the periodicity of parameters in the systems. However, if the various constituent components of the temporally nonuniform environment is with incommensurable (non-integral multiples) periods, then one has to consider the environment to be almost periodic since there is no a priori reason to expect the existence of periodic solutions. For this reason, the assumption of almost periodicity is more realistic, more important and more general when we consider the effects of the environmental factors. In fact, there have been many nice works on the positive almost periodic solutions of continuous and discrete dynamics model with almost periodic coefficients[2,3,4,5,6,7,8,9,10,11,12,13,14,15 and the references cited therein]. Zhang et al.[5] studied an almost periodic discrete multispecies Lotka-Volterra mutualism system

$$x_i(k+1) = x_i(k)\exp\left\{a_i(k) - b_i(k)x_i(k) + \sum_{j=1, j\neq i}^{n} c_{ij}(k)\frac{x_j(k)}{d_{ij} + x_j(k)}\right\}, \quad i = 1, 2, \cdots, n.$$

Sufficient conditions are obtained for the existence of a unique almost periodic solution which is globally attractive. Specially, for the discrete two-species Lotka-Volterra mutualism system, the sufficient conditions for the existence of a unique uniformly asymptotically stable almost periodic solution are obtained. Li et al.[13] studied an almost periodic discrete predator-prey models with time delays

$$\begin{cases} x(k+1) = x(k)\exp\left\{a(k) - b(k)x(k) - p(k, x(k), y(k), x(k-\mu), y(k-\nu))\dfrac{y(k)}{x(k)}\right\}, \\ y(k+1) = y(k)\exp\left\{c(k) - \dfrac{d(k)y(k)}{x(k-\mu)}\right\}. \end{cases}$$

Sufficient conditions for the permanence of the system and the existence of a unique uniformly asymptotically stable positive almost periodic sequence solution are obtained.

But to the best of the author's knowledge, to this day, still no scholars have studied the almost periodic version which is corresponding to system (1.1). Therefore, with stimulation from the works of [5,9,16,17], we will further investigate the the existence of a unique almost periodic solution of system (1.1) which is globally attractive.

Denote as Z and Z^+ the set of integers and the set of nonnegative integers, respectively. For any bounded sequence $g(n)$ defined on Z, define $g^u = \sup_{n\in Z} g(n), g^l = \inf_{n\in Z} g(n)$.

Throughout this paper, we assume that:

(H1) $b_i(k), a_{il}(k), c_{il}(k), r_j(k), d_{jl}(k)$ and $e_{jl}(k)$ are all bounded nonnegative almost periodic sequences such that

$$0 < b_i^l \leq b_i(k) \leq b_i^u, 0 < a_{il}^l \leq a_{il}(k) \leq a_{il}^u, 0 < d_{jl}^l \leq d_{jl}(k) \leq d_{jl}^u, \ \ l = 1, 2, \cdots, n;$$
$$0 < r_j^l \leq r_j(k) \leq r_j^u, 0 < c_{il}^l \leq c_{il}(k) \leq c_{il}^u, 0 < e_{jl}^l \leq e_{jl}(k) \leq e_{jl}^u, \ \ l = 1, 2, \cdots, m,$$

$i = 1, 2, \cdots, n, j = 1, 2, \cdots, m.$

From the point of view of biology, in the sequel, we assume that $\mathbf{x}(0) = (x_1(0), x_2(0), \cdots, x_n(0), y_1(0), y_2(0), \cdots, y_m(0)) > \mathbf{0}$. Then it is easy to see that, for given $\mathbf{x}(0) > \mathbf{0}$, the system (1.1) has a positive sequence solution $\mathbf{x}(k) = (x_1(k), x_2(k), \cdots, x_n(k), y_1(k), y_2(k), \cdots, y_m(k))(k \in Z^+)$ passing through $\mathbf{x}(0)$.

The remaining part of this paper is organized as follows: In Section 2, we will introduce some definitions and several useful lemmas. In Section 3, we present the permanence results for system (1.1). In Section 4, we establish the sufficient conditions for the existence of a unique globally attractive almost periodic solution of system (1.1). The main results are illustrated by an example with numerical simulation in Section 5. Finally, the conclusion ends with brief remarks in the last section.

2 Preliminaries

Firstly, we give the definitions of the terminologies involved.

Definition 2.1.[18] A sequence $x : Z \to R$ is called an almost periodic sequence if the ε-translation set of x

$$E\{\varepsilon, x\} = \{\tau \in Z :| x(n+\tau) - x(n) |< \varepsilon, \forall n \in Z\}$$

is a relatively dense set in Z for all $\varepsilon > 0$; that is, for any given $\varepsilon > 0$, there exists an integer $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon)$ contains an integer $\tau \in E\{\varepsilon, x\}$ with

$$| x(n+\tau) - x(n) |< \varepsilon, \quad \forall n \in Z.$$

τ is called an ε-translation number of $x(n)$.

Definition 2.2.[19] Let D be an open subset of R^m, $f : Z \times D \to R^m$. $f(n, x)$ is said to be almost periodic in n uniformly for $x \in D$ if for any $\varepsilon > 0$ and any compact set $S \subset D$, there exists a positive integer $l = l(\varepsilon, S)$ such that any interval of length l contains an integer τ for which

$$|f(n+\tau, x) - f(n, x)| < \varepsilon, \quad \forall (n, x) \in Z \times S.$$

τ is called an ε-translation number of $f(n, x)$.

Definition 2.3.[20] The hull of f, denoted by $H(f)$, is defined by

$$H(f) = \{g(n, x) : \lim_{k\to\infty} f(n+\tau_k, x) = g(n, x) \text{ uniformly on } Z \times S\},$$

for some sequence $\{\tau_k\}$, where S is any compact set in D.

Definition 2.4.[21] A sequence $x : Z^+ \to R$ is called an asymptotically almost periodic sequence if

$$x(n) = p(n) + q(n), \ \ \forall n \in Z^+,$$

where $p(n)$ is an almost periodic sequence and $\lim_{n\to+\infty} q(n) = 0$.

Lemma 2.5.[22] $\{x(n)\}$ is an almost periodic sequence if and only if for any integer sequence $\{k_i'\}$, there exists a subsequence $\{k_i\} \subset \{k'_i\}$ such that the sequence $\{x(n+k_i)\}$ converges uniformly for all $n \in Z$ as $i \to \infty$. Furthermore, the limit sequence is also an almost periodic sequence.

Lemma 2.6.[21] $\{x(n)\}$ is an asymptotically almost periodic sequence if and only if, for any sequence $m_i \subset Z$ satisfying $m_i > 0$ and $m_i \to \infty$ as $i \to \infty$ there exists a subsequence $\{m_{i_k}\} \subset \{m_i\}$ such that the sequence $\{x(n+m_{i_k})\}$ converges uniformly for all $n \in Z^+$ as $k \to \infty$.

3 Permanence

In this section, we establish a permanence result for system (1.1), which can be given in [1].

Proposition 3.1.[1] For every solution $(x_1(k), x_2(k), \cdots, x_n(k), y_1(k), y_2(k), \cdots, y_m(k))$ of system (1.1), we have

$$\limsup_{n\to+\infty} x_i(k) \le M_i, i = 1, 2, \cdots, n,$$

where

$$M_i = \frac{1}{a_{ii}^l} \exp\{b_i^u - 1\}. \tag{3.1}$$

Proposition 3.2.[1] Assume that

(H2) $\quad -r_j^l + \sum\limits_{l=1}^{n} d_{jl}^u M_l > 0$

holds, where $M_l(l = 1, 2, \cdots, n)$ are defined by (3.1). Then for every solution $(x_1(k), x_2(k), \cdots, x_n(k), y_1(k), y_2(k), \cdots, y_m(k))$ of system (1.1), we have

$$\limsup_{n\to+\infty} y_j(k) \le N_j, j = 1, 2, \cdots, m,$$

where

$$N_j = \frac{1}{e_{jj}^1} \exp\{-r_j^l + \sum_{l=1}^{n} d_{jl}^u M_l - 1\}. \tag{3.2}$$

Proposition 3.3.[1] In addition to (H2), assume further that

(H3) $\quad b_i^l - \sum\limits_{l=1,l\neq i}^{n} a_{il}^u M_l - \sum\limits_{l=1}^{m} c_{il}^u N_l > 0$

hold for all $i = 1, 2, \cdots, n$, where M_l, N_l are defined by (2.1) and (2.2), respectively. Then for every solution $(x_1(k), x_2(k), \cdots, x_n(k), y_1(k), y_2(k), \cdots, y_m(k))$ of system (1.1), we have

$$\liminf_{n\to+\infty} x_i(k) \ge m_i, i = 1, 2, \cdots, n,$$

where

$$m_i = \frac{b_i^l - \sum\limits_{l=1,l\neq i}^{n} a_{il}^u M_l - \sum\limits_{l=1}^{m} c_{il}^u N_l}{a_{ii}^u} \exp\{b_i^l - \sum_{l=1,l\neq i}^{n} a_{il}^u M_l - \sum_{l=1}^{m} c_{il}^u N_l\}. \tag{3.3}$$

Proposition 3.4.[1] In addition to (H2) and (H3), assume further that

(H4) $\quad -r_j^u + \sum\limits_{l=1}^{n} d_{jl}^l m_l - \sum\limits_{l=1,l\neq j}^{m} e_{jl}^u N_l > 0$

hold, where N_l and m_l are defined by (2.2) and (2.3), respectively. Then for every solution $(x_1(k), x_2(k), \cdots, x_n(k), y_1(k), y_2(k), \cdots, y_m(k))$ of system (1.1), we have

$$\liminf_{n\to+\infty} y_j(k) \geq n_j, j = 1, 2, \cdots, m,$$

where

$$n_j = \frac{-r_j^u + \sum_{l=1}^{n} d_{jl}^l m_l - \sum_{l=1,l\neq j}^{m} e_{jl}^u N_l}{e_{jj}^u} \exp\{-r_j^u + \sum_{l=1}^{n} d_{jl}^l m_l - \sum_{l=1,l\neq j}^{m} e_{jl}^u N_l\}.$$

Theorem 3.5. Assume that (H1)-(H4) hold, then system (1.1) is *permanent*.

The next result tells us that there exist solutions of system (1.1) totally in the interval of Theorem 3.5. We denote by Ω the set of all solutions $(x_1(k), x_2(k), \cdots, x_n(k), y_1(k), y_2(k), \cdots, y_m(k))$ of system (1.1) satisfying $m_i \leq x_i(k) \leq M_i, n_j \leq y_j(k) \leq N_j (i = 1, 2, \cdots, n, j = 1, 2, \cdots, m)$ for all $k \in Z^+$.

Proposition 3.6. Assume that (H1)-(H4) hold. Then $\Omega \neq \Phi$.
Proof. By the almost periodicity of $b_i(k), a_{il}(k), c_{il}(k), r_j(k), d_{jl}(k)$ and $e_{jl}(k)$, there exists an integer valued sequence $\{\delta_p\}$ with $\delta_p \to +\infty$ as $p \to +\infty$ such that

$$b_i(k+\delta_p) \to b_i(k), \;\; a_{il}(k+\delta_p) \to a_{il}(k), \;\; c_{il}(k+\delta_p) \to c_{il}(k),$$

$$r_j(k+\delta_p) \to r_j(k), \;\; d_{jl}(k+\delta_p) \to d_{jl}(k), \;\; e_{jl}(k+\delta_p) \to e_{jl}(k), \;\; \text{as } \; p \to +\infty.$$

Let ε be an arbitrary small positive number. It follows from Theorem 3.5 that there exists a positive integer N_0 such that

$$m_i - \varepsilon \leq x_i(k) \leq M_i + \varepsilon, \;\; n_j - \varepsilon \leq y_j(k) \leq N_j + \varepsilon, \;\; k > N_0.$$

Write $x_{ip}(k) = x_i(k+\delta_p)$ and $y_{jp}(k) = y_j(k+\delta_p)$ for $k \geq N_0 - \delta_p$ and $p = 1, 2, \cdots$. For any positive integer q, it is easy to see that there exists two sequences $\{x_{ip}(k) : p \geq q\}$ and $\{y_{jp}(k) : p \geq q\}$ such that the sequences $\{x_{ip}(k)\}$ and $\{y_{jp}(k)\}$ have two subsequences, respectively, denoted by $\{x_{ip}(k)\}$ and $\{y_{jp}(k)\}$ again, converging on any finite interval of **Z** as $p \to +\infty$. Thus we have two sequences $\{\widetilde{x}_i(k)\}$ and $\{\widetilde{y}_j(k)\}$ such that

$x_{ip}(k) \to \widetilde{x}_i(k), \;\; y_{jp}(k) \to \widetilde{y}_j(k)$ for $k \in$ **Z** as $p \to +\infty$.

This, combined with

$$x_i(k+1+\delta_p) = x_i(k+\delta_p)\exp\left[b_i(k+\delta_p) - \sum_{l=1}^{n} a_{il}(k+\delta_p)x_l(k+\delta_p) - \sum_{l=1}^{m} c_{il}(k+\delta_p)y_l(k+\delta_p)\right],$$

$$y_j(k+1+\delta_p) = y_j(k+\delta_p)\exp\left[-r_j(k+\delta_p) + \sum_{l=1}^{n} d_{jl}(k+\delta_p)x_l(k+\delta_p) - \sum_{l=1}^{m} e_{jl}(k+\delta_p)y_l(k+\delta_p)\right],$$

$$i = 1, 2, \cdots, n, \;\; j = 1, 2, \cdots, m$$

gives us

$$\widetilde{x}_i(k+1) = \widetilde{x}_i(k)\exp\left[b_i(k) - \sum_{l=1}^{n} a_{il}(k)\widetilde{x}_l(k) - \sum_{l=1}^{m} c_{il}(k)\widetilde{y}_l(k)\right],$$

$$\widetilde{y}_j(k+1) = \widetilde{y}_j(k)\exp\left[-r_j(k) + \sum_{l=1}^{n} d_{jl}(k)\widetilde{x}_l(k) - \sum_{l=1}^{m} e_{jl}(k)\widetilde{y}_l(k)\right],$$

$$i = 1, 2, \cdots, n, \;\; j = 1, 2, \cdots, m.$$

We can easily see that $(\widetilde{x}_1(k), \widetilde{x}_2(k), \cdots, \widetilde{x}_n(k), \widetilde{y}_1(k), \widetilde{y}_2(k), \cdots, \widetilde{y}_m(k))$ is a solution of system (1.1) and $m_i - \varepsilon \leq \widetilde{x}_i(k) \leq M_i + \varepsilon, \;\; n_j - \varepsilon \leq \widetilde{y}_j(k) \leq N_j + \varepsilon$ for $k \in$ **Z**. Since ε is an arbitrary small positive number, it follows that $m_i \leq \widetilde{x}_i(k) \leq M_i, \;\; n_j \leq \widetilde{y}_j(k) \leq N_j$ and hence we complete the proof.

4 Almost Periodic Solution

The main result of this paper concerns the existence of a unique global attractive almost periodic solution of system (1.1).

Theorem 4.1. Assume that (H1)-(H4) and

(H5) $\rho_i = \max\{|1 - a^l_{ii} m_i|,\ |1 - a^u_{ii} M_i|\} + \sum\limits_{l=1, l\neq i}^{n} a^u_{il} M_l + \sum\limits_{l=1}^{m} c^u_{il} N_l < 1,\ i = 1, 2, \cdots, n,$

$$\sigma_j = \max\{|1 - e^l_{jj} n_j|,\ |1 - e^u_{jj} N_j|\} + \sum_{l=1, l\neq i}^{m} e^u_{jl} N_l + \sum_{l=1}^{n} d^u_{jl} M_l < 1,\ j = 1, 2, \cdots, m,$$

hold. Then system (1.1) admits a unique almost periodic solution which is globally attractive.

Proof. It follows from Proposition 3.6 that there exists a solution $(x_1(k), x_2(k), \cdots, x_n(k), y_1(k), y_2(k), \cdots, y_m(k))$ of system (1.1) satisfying $m_i \le x_i(k) \le M_i, n_j \le y_j(k) \le N_j, k \in Z^+ (i = 1, 2, \cdots, n, j = 1, 2, \cdots, m)$. Let $\{\delta_k\}$ be any integer valued sequence such that $\delta_k \to +\infty$ as $k \to +\infty$. Using the Mean Value Theorem, for $p \neq q$, we get

$$\begin{aligned} \ln x_i(k+\delta_p) - \ln x_i(k+\delta_q) &= \frac{1}{\xi_i(k,p,q)}[x_i(k+\delta_p) - x_i(k+\delta_q)], \\ \ln y_j(k+\delta_p) - \ln y_j(k+\delta_q) &= \frac{1}{\eta_j(k,p,q)}[y_j(k+\delta_p) - y_j(k+\delta_q)], \end{aligned} \quad (4.1)$$

where $\xi_i(k,p,q)$ lies between $x_i(k+\delta_p)$ and $x_i(k+\delta_q)$, and $\eta_j(k,p,q)$ lies between $y_j(k+\delta_p)$ and $y_j(k+\delta_q)$. Then

$$\begin{aligned} &|x_i(k+\delta_p) - x_i(k+\delta_q)| \le M_i |\ln x_i(k+\delta_p) - \ln x_i(k+\delta_q)|, \\ &|y_j(k+\delta_p) - y_j(k+\delta_q)| \le N_j |\ln y_j(k+\delta_p) - \ln y_j(k+\delta_q)|, \quad k \in Z^+. \end{aligned} \quad (4.2)$$

For convenience, we introduce $\Lambda_s(k, \delta_p, \delta_q)$ through

$$\Lambda_s(k, \delta_p, \delta_q) = \begin{cases} \varphi_s(k, \delta_p, \delta_q), & 1 \le s \le n, \\ \psi_{s-n}(k, \delta_p, \delta_q), & n+1 \le s \le n+m,\ k \in Z^+,\ \delta_p > 0,\ \delta_q > 0, \end{cases} \quad (4.3)$$

where

$$\begin{aligned} \varphi_i(k, \delta_p, \delta_q) &= |\ln x_i(k+\delta_p) - \ln x_i(k+\delta_q)|, \quad s = i = 1, 2, \cdots, n, \\ \psi_j(k, \delta_p, \delta_q) &= |\ln y_j(k+\delta_p) - \ln y_j(k+\delta_q)|, \quad s - n = j = 1, 2, \cdots, m. \end{aligned}$$

Thus, when $1 \le s \le n$, we have

$$\begin{aligned}
&\Lambda_s(k+1,\delta_p,\delta_q) = \varphi_i(k+1,\delta_p,\delta_q) = |\ln x_i(k+1+\delta_p) - \ln x_i(k+1+\delta_q)| \\
&= \Bigg| \ln x_i(k+\delta_p) - \ln x_i(k+\delta_q) \\
&\quad + b_i(k+\delta_p) - \sum_{l=1}^{n} a_{il}(k+\delta_p)x_l(k+\delta_p) - \sum_{l=1}^{m} c_{il}(k+\delta_p)y_l(k+\delta_p) \\
&\quad - b_i(k+\delta_q) + \sum_{l=1}^{n} a_{il}(k+\delta_q)x_l(k+\delta_q) + \sum_{l=1}^{m} c_{il}(k+\delta_q)y_l(k+\delta_q) \Bigg| \\
&\le \Bigg| \ln x_i(k+\delta_p) - \ln x_i(k+\delta_q) - a_{ii}(k+\delta_p)[x_i(k+\delta_p) - x_i(k+\delta_q)] \Bigg| \\
&\quad + \Bigg| b_i(k+\delta_p) - b_i(k+\delta_q) \Bigg| + \Bigg| [a_{ii}(k+\delta_q) - a_{ii}(k+\delta_p)]x_i(k+\delta_q) \Bigg| \\
&\quad + \sum_{l=1,l\ne i}^{n} \Bigg| a_{il}(k+\delta_p)\big[x_l(k+\delta_p) - x_l(k+\delta_q)\big] \Bigg| \\
&\quad + \sum_{l=1,l\ne i}^{n} \Bigg| \big[a_{il}(k+\delta_p) - a_{il}(k+\delta_q)\big] x_l(k+\delta_q) \Bigg| \\
&\quad + \sum_{l=1}^{m} \Bigg| c_{il}(k+\delta_p)\big[y_l(k+\delta_p) - y_l(k+\delta_q)\big] \Bigg| \\
&\quad + \sum_{l=1}^{m} \Bigg| \big[c_{il}(k+\delta_p) - c_{il}(k+\delta_q)\big] y_l(k+\delta_q) \Bigg|.
\end{aligned} \tag{4.4}$$

When $n+1 \le s \le n+m$, we have

$$\begin{aligned}
&\Lambda_s(k+1,\delta_p,\delta_q) = \psi_j(k+1,\delta_p,\delta_q) = |\ln y_j(k+1+\delta_p) - \ln y_j(k+1+\delta_q)| \\
&= \Bigg| \ln y_j(k+\delta_p) - \ln y_j(k+\delta_q) \\
&\quad - r_j(k+\delta_p) + \sum_{l=1}^{n} d_{jl}(k+\delta_p)x_l(k+\delta_p) - \sum_{l=1}^{m} e_{jl}(k+\delta_p)y_l(k+\delta_p) \\
&\quad + r_j(k+\delta_q) - \sum_{l=1}^{n} d_{jl}(k+\delta_q)x_l(k+\delta_q) + \sum_{l=1}^{m} e_{jl}(k+\delta_q)y_l(k+\delta_q) \Bigg| \\
&\le \Bigg| \ln y_j(k+\delta_p) - \ln y_j(k+\delta_q) - e_{jj}(k+\delta_p)[y_j(k+\delta_p) - y_j(k+\delta_q)] \Bigg| \\
&\quad + \Bigg| r_j(k+\delta_p) - r_j(k+\delta_q) \Bigg| + \Bigg| [e_{jj}(k+\delta_q) - e_{jj}(k+\delta_p)]y_j(k+\delta_q) \Bigg| \\
&\quad + \sum_{l=1,l\ne i}^{m} \Bigg| e_{jl}(k+\delta_p)\big[y_l(k+\delta_p) - y_l(k+\delta_q)\big] \Bigg| \\
&\quad + \sum_{l=1,l\ne i}^{m} \Bigg| \big[e_{jl}(k+\delta_p) - e_{jl}(k+\delta_q)\big] y_l(k+\delta_q) \Bigg|
\end{aligned}$$

$$+\sum_{l=1}^{n}\Big|d_{jl}(k+\delta_p)\big[x_l(k+\delta_p)-x_l(k+\delta_q)\big]\Big|$$
$$+\sum_{l=1}^{n}\Big|\big[d_{jl}(k+\delta_p)-d_{jl}(k+\delta_q)\big]x_l(k+\delta_q)\Big|. \tag{4.5}$$

Let ε_1 be an arbitrary positive number. By the almost periodicity of $\{b_i(k)\}, \{a_{il}(k)\}, \{c_{il}(k)\}, \{r_j(k)\}, \{d_{jl}(k)\}$ and $\{e_{jl}(k)\}$ and the boundedness of $\{(x_1(k), x_2(k), \cdots, x_n(k), y_1(k), y_2(k), \cdots, y_m(k))\}$, it follows from Lemmas 2.2 and 2.4 that there exists a positive integer $K_1 = K_1(\varepsilon_1)$ such that, for any $\delta_q \geq \delta_p \geq K_1$ and $k \in Z^+$(if necessary, we can choose subsequences of $\{\delta_p\}$ and $\{\delta_q\}$),

$$\Big|b_i(k+\delta_p)-b_i(k+\delta_q)\Big|<\frac{\varepsilon_1}{4},\quad \Big|[a_{ii}(k+\delta_q)-a_{ii}(k+\delta_p)]x_i(k+\delta_q)\Big|<\frac{\varepsilon_1}{4},$$
$$\sum_{l=1,l\neq i}^{n}\Big|a_{il}(k+\delta_p)\big[x_l(k+\delta_p)-x_l(k+\delta_q)\big]\Big|<\frac{\varepsilon_1}{4},$$
$$\sum_{l=1}^{m}\Big|\big[c_{il}(k+\delta_p)-c_{il}(k+\delta_q)\big]y_l(k+\delta_q)\Big|<\frac{\varepsilon_1}{4},$$
$$\Big|r_j(k+\delta_p)-r_j(k+\delta_q)\Big|<\frac{\varepsilon_1}{4},\quad \Big|[e_{jj}(k+\delta_q)-e_{jj}(k+\delta_p)]y_j(k+\delta_q)\Big|<\frac{\varepsilon_1}{4},$$
$$\sum_{l=1,l\neq i}^{m}\Big|\big[e_{jl}(k+\delta_p)-e_{jl}(k+\delta_q)\big]y_l(k+\delta_q)\Big|<\frac{\varepsilon_1}{4},$$
$$\sum_{l=1}^{n}\Big|\big[d_{jl}(k+\delta_p)-d_{jl}(k+\delta_q)\big]x_l(k+\delta_q)\Big|<\frac{\varepsilon_1}{4}. \tag{4.6}$$

It follows from (4.1) and (4.3)-(4.6) that, for $k \in Z^+$ and $\delta_q \geq \delta_p \geq K_1$,

$$\begin{aligned}
\varphi_i(k+1,\delta_p,\delta_q) &< \Big|1-a_{ii}(k+\delta_p)\xi_i(k,p,q)\Big|\varphi_i(k,\delta_p,\delta_q)\\
&\quad+\sum_{l=1,l\neq i}^{n}\Big|a_{il}(k+\delta_p)\xi_l(k,p,q)\Big|\varphi_l(k,\delta_p,\delta_q)\\
&\quad+\sum_{l=1}^{m}\Big|c_{il}(k+\delta_p)\eta_l(k,p,q)\Big|\psi_l(k,\delta_p,\delta_q)+\varepsilon_1\\
&\leq \rho_i\max\{\varphi_i(k,\delta_p,\delta_q),\psi_j(k,\delta_p,\delta_q)\}+\varepsilon_1,\\
\psi_j(k+1,\delta_p,\delta_q) &< \Big|1-e_{jj}(k+\delta_p)\eta_j(k,p,q)\Big|\psi_j(k,\delta_p,\delta_q)\\
&\quad+\sum_{l=1,l\neq i}^{m}\Big|e_{jl}(k+\delta_p)\eta_l(k,p,q)\Big|\psi_l(k,\delta_p,\delta_q)\\
&\quad+\sum_{l=1}^{n}\Big|d_{jl}(k+\delta_p)\xi_l(k,p,q)\Big|\varphi_l(k,\delta_p,\delta_q)+\varepsilon_1\\
&\leq \sigma_j\max\{\varphi_i(k,\delta_p,\delta_q),\psi_j(k,\delta_p,\delta_q)\}+\varepsilon_1.
\end{aligned}$$

Then

$$\begin{aligned}
&\varphi_i(k,\delta_p,\delta_q)<\rho_i\max\{\varphi_i(k-1,\delta_p,\delta_q),\psi_j(k-1,\delta_p,\delta_q)\}+\varepsilon_1,\\
&\varphi_i(k-1,\delta_p,\delta_q)<\rho_i\max\{\varphi_i(k-2,\delta_p,\delta_q),\psi_j(k-2,\delta_p,\delta_q)\}+\varepsilon_1,\\
&\cdots\cdots\cdots\cdots,\\
&\varphi_i(1,\delta_p,\delta_q)<\rho_i\max\{\varphi_i(0,\delta_p,\delta_q),\psi_j(0,\delta_p,\delta_q)\}+\varepsilon_1.
\end{aligned}$$

And we have

$$\varphi_i(k,\delta_p,\delta_q) < \rho_i^k \max\{\varphi_i(0,\delta_p,\delta_q),\psi_j(0,\delta_p,\delta_q)\} + \frac{1-\rho_i^k}{1-\rho_i}\varepsilon_1, \tag{4.7}$$

for $k \in Z^+$ and $\delta_q \geq \delta_q \geq K_1$.

By a similar argument as that in (4.7), we could easily obtain that

$$\psi_j(k,\delta_p,\delta_q) < \sigma_j^k \max\{\varphi_i(0,\delta_p,\delta_q),\psi_j(0,\delta_p,\delta_q)\} + \frac{1-\sigma_j^k}{1-\sigma_j}\varepsilon_1,$$

Since $\rho_i < 1$ and $\sigma_j < 1$, for arbitrary $\varepsilon > 0$, there exists a positive integer $K = K(\varepsilon) > K_1$ such that, for any $\delta_q \geq \delta_p \geq K$,

$$\varphi_i(k,\delta_p,\delta_q) < \frac{\varepsilon}{\max\limits_{1\leq i\leq n}\{M_i\}},\ \ \psi_j(k,\delta_p,\delta_q) < \frac{\varepsilon}{\max\limits_{1\leq j\leq m}\{N_j\}}$$

for $k \in Z^+$.

This combined with (4.2) gives us

$$\left|x_i(k+\delta_p) - x_i(k+\delta_q)\right| < \varepsilon,\ \ \left|y_j(k+\delta_p) - y_j(k+\delta_q)\right| < \varepsilon.$$

for $k \in Z^+$ and $\delta_q \geq \delta_q \geq K$. It follows from Lemma 2.6 that the sequence $\{(x_1(k), x_2(k), \cdots, x_n(k), y_1(k), y_2(k), \cdots, y_m(k))\}$ is asymptotically almost periodic. Thus, by Definition 2.4, we can express it as

$$x_i(k) = p_i(k) + q_i(k),\ \ y_j(k) = u_j(k) + v_j(k), \tag{4.8}$$

$i = 1, 2, \cdots, n, j = 1, 2, \cdots, m$, where $\{p_i(k)\}$ and $\{u_j(k)\}$ are almost periodic in $k \in Z$ and $q_i(k) \to 0$, $v_j(k) \to 0$ as $k \to +\infty$. In the following we show that $\{(p_1(k), p_2(k), \cdots, p_n(k), u_1(k), u_2(k), \cdots, u_m(k))\}$ is an almost periodic solution of system (1.1).

Define

$$F_s(k) = \begin{cases} f_s(k), & 1 \leq s \leq n, \\ \tilde{f}_{s-n}(k), & n+1 \leq s \leq n+m, \end{cases}$$

and

$$G_s(k) = \begin{cases} g_s(k), & 1 \leq s \leq n, \\ \tilde{g}_{s-n}(k), & n+1 \leq s \leq n+m, \end{cases}$$

where

$$\begin{aligned}
f_i(k) &= b_i(k) - \sum_{l=1}^{n} a_{il}(k)(p_l(k)+q_l(k)) - \sum_{l=1}^{m} c_{il}(k)(u_l(k)+v_l(k)),\ \ s = i = 1,2,\cdots,n, \\
\tilde{f}_j(k) &= -r_j(k) + \sum_{l=1}^{n} d_{jl}(k)(p_l(k)+q_l(k)) - \sum_{l=1}^{m} e_{jl}(k)(u_l(k)+v_l(k)),\ \ s-n = j = 1,2,\cdots,m, \\
g_i(k) &= b_i(k) - \sum_{l=1}^{n} a_{il}(k)p_l(k) - \sum_{l=1}^{m} c_{il}(k)u_l(k),\ \ s = i = 1,2,\cdots,n, \\
\tilde{g}_j(k) &= -r_j(k) + \sum_{l=1}^{n} d_{jl}(k)p_l(k) - \sum_{l=1}^{m} e_{jl}(k)u_l(k),\ \ s-n = j = 1,2,\cdots,m,
\end{aligned}$$

It follows from (1.1), (4.8) and the Mean Value Theorem that

$$\begin{aligned}
& p_i(k+1)+q_i(k+1) \\
&= [p_i(k)+q_i(k)]\exp\{f_i(k)\} \\
&= p_i(k)[\exp\{f_i(k)\}-\exp\{g_i(k)\}]+p_i(k)\exp\{g_i(k)\}+q_i(k)\exp\{f_i(k)\} \\
&= -p_i(k)\exp\{\xi_i(k)\}\left[\sum_{l=1}^{n}a_{il}(k)q_l(k)+\sum_{l=1}^{m}c_{il}(k)v_l(k)\right] \\
&\quad + p_i(k)\exp\{g_i(k)\}+q_i(k)\exp\{f_i(k)\},
\end{aligned}$$

$$\begin{aligned}
& u_j(k+1)+v_j(k+1) \\
&= [u_j(k)+v_j(k)]\exp\{\tilde{f}_j(k)\} \\
&= u_j(k)[\exp\{\tilde{f}_j(k)\}-\exp\{\tilde{g}_j(k)\}]+u_j(k)\exp\{\tilde{g}_j(k)\}+v_j(k)\exp\{\tilde{f}_j(k)\} \\
&= u_j(k)\exp\{\eta_j(k)\}\left[\sum_{l=1}^{n}d_{jl}(k)q_l(k)-\sum_{l=1}^{m}e_{jl}(k)v_l(k)\right] \\
&\quad + u_j(k)\exp\{\tilde{g}_j(k)\}+v_j(k)\exp\{\tilde{f}_j(k)\},
\end{aligned}$$

where $\xi_i(k) = \theta_i(k)f_i(k) + (1-\theta_i(k))g_i(k)$ and $\eta_j(k) = \gamma_j(k)\tilde{f}_j(k) + (1-\gamma_j(k))\tilde{g}_j(k)$ for some $\theta_i(k), \gamma_j(k) \in [0,1]$. Thus

$$\begin{aligned}
& p_i(k+1)-p_i(k)\exp\{g_i(k)\} \\
&= -p_i(k)\exp\{\xi_i(k)\}\left[\sum_{l=1}^{n}a_{il}(k)q_l(k)+\sum_{l=1}^{m}c_{il}(k)v_l(k)\right]-q_i(k+1)+q_i(k)\exp\{f_i(k)\}, \\
& u_j(k+1)-u_j(k)\exp\{\tilde{g}_j(k)\} \\
&= u_j(k)\exp\{\eta_j(k)\}\left[\sum_{l=1}^{n}d_{jl}(k)q_l(k)-\sum_{l=1}^{m}e_{jl}(k)v_l(k)\right]+v_j(k)\exp\{\tilde{f}_j(k)\}.
\end{aligned}$$

Let

$$V_i(k) = p_i(k+1)-p_i(k)\exp\{g_i(k)\},$$

and

$$U_i(k) = u_j(k+1)-u_j(k)\exp\{\tilde{g}_j(k)\}.$$

By the boundedness of the almost periodic sequences $\{a_{il}(k)\}$,$\{c_{il}(k)\}$,$\{d_{jl}(k)\}$,$\{e_{jl}(k)\}$ and the fact that $q_i(k)\to 0, v_j(k)\to 0$ as $k\to+\infty$, we obtain
$V_i(k)\to 0$, $U_j(k)\to 0$ as $k\to+\infty$.
We claim that $V_i(k)\equiv 0$ and $U_j(k)\equiv 0$. Otherwise, there exists an integer $k_0\in Z$ such that $V_i(k_0)\neq 0$. By the almost periodicity of $\{b_i(k)\}$,$\{a_{il}(k)\}$,$\{c_{il}(k)\}$ and $\{p_i(k)\}$, there exists an integer valued sequence τ_p such that $\tau_p\to+\infty$ as $p\to+\infty$ and

$$b_i(k+\tau_p)\to b_i(k),\ a_{il}(k+\tau_p)\to a_{il}(k),\ c_{il}(k+\tau_p)\to c_{il}(k),\ p_i(k+\tau_p)\to p_i(k)$$

uniformly for all $k\in \mathbf{Z}^+$. Then we have

$$\begin{aligned}
V_i(k_0+\tau_p) &= p_i(k_0+\tau_p+1)-p_i(k_0+\tau_p)\exp\{g_i(k_0+\tau_p)\} \\
&\to p_i(k_0+1)-p_i(k_0)\exp\{g_i(k_0)\} \\
&= V_i(k_0)
\end{aligned}$$

as $p\to+\infty$, which contradicts that $V_i(k)\to 0$ as $k\to+\infty$. This proves the claim. Hence

$$p_i(k+1) = p_i(k)\exp\{g_i(k)\}. \tag{4.9}$$

By a similar argument as that in (4.9), we could obtain that

$$u_j(k+1) = u_j(k)\exp\{\tilde{g}_j(k)\};$$

that is, $\{(p_1(k), p_2(k), \cdots, p_n(k), u_1(k), u_2(k), \cdots, u_m(k))\}$ is an almost periodic solution of system (1.1).

Then, we prove that almost periodic solution $\{(p_1(k), p_2(k), \cdots, p_n(k), u_1(k), u_2(k), \cdots, u_m(k))\}$ is globally attractive. The proof is similar to the proof of Theorem 2 in [1]. However, for the sake of completeness, here we give the complete proof.

Assume that $\{(x_1(k), x_2(k), \cdots, x_n(k), y_1(k), y_2(k), \cdots, y_m(k))\}$ is a solution of system (1.1) satisfying (H1)-(H4). Let

$$x_i(k) = p_i(k)\exp\{\alpha_i(k)\}, \quad i = 1, 2, \cdots, n,$$

$$y_j(k) = u_j(k)\exp\{\beta_j(k)\}, \quad j = 1, 2, \cdots, m.$$

Then system (1.1) is equivalent to

$$\begin{aligned}
\alpha_i(k+1) &= \ln x_i(k+1) - \ln p_i(k+1) \\
&= \ln x_i(k) + b_i(k) - \sum_{l=1}^{n} a_{il}(k)x_l(k) - \sum_{l=1}^{m} c_{il}(k)y_l(k) \\
&\quad - \ln p_i(k) - b_i(k) + \sum_{l=1}^{n} a_{il}(k)p_l(k) + \sum_{l=1}^{m} c_{il}(k)u_l(k) \\
&= \alpha_i(k) - a_{ii}(k)[x_i(k) - p_i(k)] - \sum_{l=1, l\neq i}^{n} a_{il}(k)[x_l(k) - p_l(k)] - \sum_{l=1}^{m} c_{il}(k)[y_l(k) - u_l(k)] \\
&= \alpha_i(k) - a_{ii}(k)p_i(k)[\exp\{\alpha_i(k)\} - 1] - \sum_{l=1, l\neq i}^{n} a_{il}(k)p_l(k)[\exp\{\alpha_l(k)\} - 1] \\
&\quad - \sum_{l=1}^{m} c_{il}(k)u_l(k)[\exp\{\beta_l(k)\} - 1], \quad i = 1, 2, \cdots, n,
\end{aligned}$$

$$\begin{aligned}
\beta_j(k+1) &= \ln y_j(k+1) - \ln u_j(k+1) \\
&= \ln y_j(k) - r_j(k) + \sum_{l=1}^{n} d_{jl}(k)x_l(k) - \sum_{l=1}^{m} e_{jl}(k)y_l(k) \\
&\quad - \ln u_j(k) + r_j(k) - \sum_{l=1}^{n} d_{jl}(k)p_l(k) + \sum_{l=1}^{m} e_{jl}(k)u_l(k) \\
&= \beta_j(k) - e_{jj}(k)[y_j(k) - u_j(k)] - \sum_{l=1, l\neq i}^{m} e_{il}(k)[y_l(k) - u_l(k)] + \sum_{l=1}^{n} d_{jl}(k)[x_l(k) - p_l(k)] \\
&= \beta_j(k) - e_{jj}(k)u_j(k)[\exp\{\beta_j(k)\} - 1] - \sum_{l=1, l\neq i}^{m} e_{il}(k)u_l(k)[\exp\{\beta_l(k)\} - 1] \\
&\quad + \sum_{l=1}^{n} d_{jl}(k)p_l(k)[\exp\{\alpha_l(k)\} - 1], \quad j = 1, 2, \cdots, m.
\end{aligned}$$

Therefore,

$$\begin{aligned}\alpha_i(k+1) &= \alpha_i(k)[1-a_{ii}(k)p_i(k)\exp\{\lambda_i(k)\alpha_i(k)\}] - \sum_{l=1,l\neq i}^{n} a_{il}(k)p_l(k)\alpha_l(k)\exp\{\lambda_l(k)\alpha_l(k)\} \\ &\quad - \sum_{l=1}^{m} c_{il}(k)u_l(k)\beta_l(k)\exp\{\overline{\lambda_l}(k)\beta_l(k)\}, \quad i=1,2,\cdots,n, \quad (4.10)\\ \beta_j(k+1) &= \beta_j(k)[1-e_{jj}(k)u_j(k)\exp\{\overline{\lambda_j}(k)\beta_j(k)\}] - \sum_{l=1,l\neq i}^{m} e_{il}(k)u_l(k)\beta_l(k)\exp\{\overline{\lambda_l}(k)\beta_l(k)\} \\ &\quad + \sum_{l=1}^{n} d_{il}(k)p_l(k)\alpha_l(k)\exp\{\lambda_l(k)\alpha_l(k)\}, \quad j=1,2,\cdots,m. \quad (4.11)\end{aligned}$$

where $\lambda_i(k), \overline{\lambda_j}(k) \in [0,1]$. To complete the proof, it suffices to show that

$$\lim_{k\to+\infty} \alpha_i(k) = 0, \quad i=1,2,\cdots,n, \quad (4.12)$$

$$\lim_{k\to+\infty} \beta_j(k) = 0, \quad j=1,2,\cdots,m. \quad (4.13)$$

In view of (H5), we can choose $\varepsilon > 0$ such that

$$\rho_i^{\varepsilon} = \max\{|1-a_{ii}^l(m_i-\varepsilon)|,\ |1-a_{ii}^u(M_i+\varepsilon)|\} + \sum_{l=1,l\neq i}^{n} a_{il}^u(M_l+\varepsilon) + \sum_{l=1}^{m} c_{il}^u(N_l+\varepsilon) < 1,$$

$$\sigma_j^{\varepsilon} = \max\{|1-e_{jj}^l(n_j-\varepsilon)|,\ |1-e_{jj}^u(N_j+\varepsilon)|\} + \sum_{l=1,l\neq i}^{m} e_{jl}^u(N_l+\varepsilon) + \sum_{l=1}^{n} d_{jl}^u(M_l+\varepsilon) < 1,$$

$i=1,2,\cdots,n, j=1,2,\cdots,m.$

Let $\rho = \max\{\rho_i^{\varepsilon}\}$ and $\sigma = \max\{\sigma_j^{\varepsilon}\}$, then $\rho < 1$ and $\sigma < 1$. According to Theorem 3.5, there exists a positive integer $k_0 \in Z^+$ such that

$$m_i - \varepsilon \le x_i(k) \le M_i + \varepsilon, \quad m_i - \varepsilon \le p_i(k) \le M_i + \varepsilon, \quad i=1,2,\cdots,n,$$

$$n_j - \varepsilon \le y_j(k) \le N_j + \varepsilon, \quad n_j - \varepsilon \le u_j(k) \le N_j + \varepsilon, \quad j=1,2,\cdots,m$$

for $k \ge k_0$.

Notice that $\lambda_i(k) \in [0,1]$ implies that $p_i(k)\exp\{\lambda_i(k)\alpha_i(k)\}$ lies between $p_i(k)$ and $x_i(k)$, $\overline{\lambda_j}(k) \in [0,1]$ implies that $u_j(k)\exp\{\overline{\lambda_j}(k)\beta_j(k)\}$ lies between $u_j(k)$ and $y_j(k)$. From (4.10) and (4.11), we get

$$\begin{aligned}|\alpha_i(k+1)| &\le \max\{|1-a_{ii}^l(m_i-\varepsilon)|,\ |1-a_{ii}^u(M_i+\varepsilon)|\}|\alpha_i(k)| + \sum_{l=1,l\neq i}^{n} a_{il}^u(M_l+\varepsilon)|\alpha_l(k)| \\ &\quad + \sum_{l=1}^{m} c_{il}^u(N_l+\varepsilon)|\beta_l(k)|,\ i=1,2,\cdots,n, \quad (4.14)\\ |\beta_j(k+1)| &\le \max\{|1-e_{jj}^l(n_j-\varepsilon)|,\ |1-e_{jj}^u(N_j+\varepsilon)|\}|\beta_j(k)| + \sum_{l=1,l\neq i}^{m} e_{jl}^u(N_l+\varepsilon)|\beta_l(k)| \\ &\quad + \sum_{l=1}^{n} d_{jl}^u(M_l+\varepsilon)|\alpha_l(k)|,\ j=1,2,\cdots,m, \quad (4.15)\end{aligned}$$

for $k \ge k_0$.

In view of (4.14) and (4.15), we get

$$\max_{1\le i\le n}|\alpha_i(k+1)| \le \rho \max_{1\le i\le n, 1\le j\le m}\{|\alpha_i(k)|, |\beta_j(k)|\},$$

$$\max_{1\le i\le n}|\beta_j(k+1)| \le \sigma \max_{1\le i\le n, 1\le j\le m}\{|\alpha_i(k)|, |\beta_j(k)|\}, \quad k \ge k_0.$$

This implies

$$\max_{1\le i\le n}|\alpha_i(k)| \le \rho^{k-k_0} \max_{1\le i\le n, 1\le j\le m}\{|\alpha_i(k)|, |\beta_j(k)|\},$$

$$\max_{1\le i\le n}|\beta_j(k)| \le \sigma^{k-k_0} \max_{1\le i\le n, 1\le j\le m}\{|\alpha_i(k)|, |\beta_j(k)|\}, \quad k \ge k_0.$$

Then (4.12) and (4.13) hold. So, we can obtain

$$\lim_{k\to+\infty}|x_i(k) - p_i(k)| = 0, \;\; i = 1, 2, \cdots, n, \tag{4.16}$$

$$\lim_{k\to+\infty}|y_i(k) - u_i(k)| = 0, \;\; j = 1, 2, \cdots, m. \tag{4.17}$$

Now, we show that there is only one positive almost periodic solution of system (1.1). For any two positive almost periodic solutions $(p_1(k), p_2(k), \cdots, p_n(k), u_1(k), u_2(k), \cdots, u_m(k))$ and $(z_1(k), z_2(k), \cdots, z_n(k), w_1(k), w_2(k), \cdots, w_m(k))$ of system (1.1), we claim that $p_i(k) = z_i(k)$, $u_j(k) = w_j(k)(i = 1, 2, \cdots, n, j = 1, 2, \cdots, m)$ for all $k \in \mathbf{Z}^+$. Otherwise there must be at least one positive integer $K^* \in \mathbf{Z}^+$ such that $p_i(K^*) \ne z_i(K^*)$ or $u_j(K^*) \ne w_j(K^*)$ for a certain positive integer i or j, i.e., $\Omega_1 = |p_i(K^*) - z_i(K^*)| > 0$ or $\Omega_2 = |u_j(K^*) - w_j(K^*)| > 0$. So we can easily know that

$$\begin{aligned}\Omega_1 &= |\lim_{p\to+\infty} p_i(K^*+\delta_p) - \lim_{p\to+\infty} z_i(K^*+\delta_p)| = \lim_{p\to+\infty}|p_i(K^*+\delta_p) - z_i(K^*+\delta_p)| \\ &= \lim_{k\to+\infty}|p_i(k) - z_i(k)| > 0,\end{aligned}$$

or

$$\begin{aligned}\Omega_2 &= |\lim_{p\to+\infty} u_j(K^*+\delta_p) - \lim_{p\to+\infty} w_j(K^*+\delta_p)| = \lim_{p\to+\infty}|u_j(K^*+\delta_p) - w_j(K^*+\delta_p)| \\ &= \lim_{k\to+\infty}|u_j(k) - w_j(k)| > 0,\end{aligned}$$

which is a contradiction to (4.16) or (4.17). Thus $p_i(k) = z_i(k)$, $u_j(k) = w_j(k)(i = 1, 2, \cdots, n, j = 1, 2, \cdots, m)$ hold for $\forall k \in \mathbf{Z}^+$. Therefore, system (1.1) admits a unique almost periodic solution which is globally attractive. This completes the proof of Theorem 4.1. ☐

5 Numerical Simulations

In this section, we give the following example to check the feasibility of our result.
Example Consider the following discrete Lotka-Volterra competition predator-prey system:

$$\left\{\begin{aligned} x_1(k+1) &= x_1(k)\exp\Big\{1.2 - 0.02\sin(\sqrt{2}k) - (1.05 + 0.01\sin(\sqrt{3}k))x_1(k) \\ &\qquad - (0.025 + 0.002\cos(\sqrt{5}k))y_1(k) - (0.02 + 0.001\cos(\sqrt{2}k))y_2(k)\Big\}, \\ y_1(k+1) &= y_1(k)\exp\Big\{ - 0.01 - 0.025\cos(\sqrt{3}k) + (1.02 + 0.003\sin(\sqrt{2}k))x_1(k) \\ &\qquad - (1.08 + 0.015\sin(\sqrt{2}k))y_1(k) - (0.025 + 0.002\cos(\sqrt{5}k))y_2(k)\Big\}, \\ y_2(k+1) &= y_2(k)\exp\Big\{ - 0.015 - 0.03\sin(\sqrt{5}k) + (1.03 + 0.0025\cos(\sqrt{2}k))x_1(k) \\ &\qquad - (0.028 + 0.0015\cos(\sqrt{2}k))y_1(k) - (1.1 + 0.02\sin(\sqrt{2}n))y_2(k)\Big\}. \end{aligned}\right. \tag{5.1}$$

A computation shows that

$$m_1 \approx 0.9846,\ M_1 \approx 1.1739,\ m_2 \approx 0.9072,\ M_2 \approx 1.1138,\ m_3 \approx 0.8912,\ M_3 \approx 1.2794,$$

and moreover, we have

$$\rho_1 \approx 0.1842,\ \rho_2 \approx 0.0174,\ \rho_3 \approx 0.1246,$$

that $\max\{\rho_1, \rho_2, \rho_3\} < 1$. It is easy to see that the condition (H5) are satisfied. Hence, there exists a unique globally attractive almost periodic solution of system (5.1). Our numerical simulations support our results(see Figs.1,2 and 3).

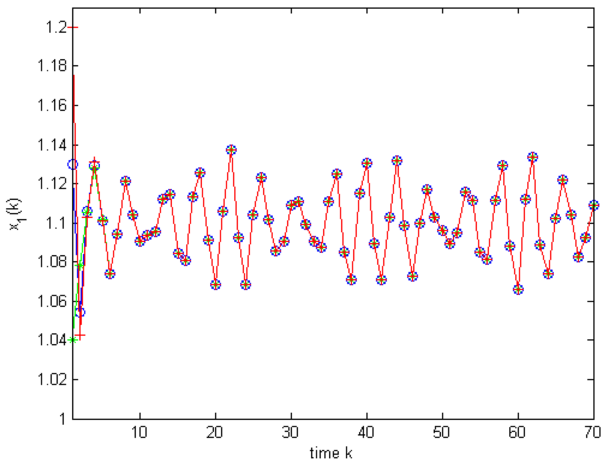

Figure1: Dynamic behavior of $x_1(k)$ of the solution $(x_1(k), y_1(k), y_2(k))$ to system (5.1) with the initial conditions (1.13,1.17,1.2), (1.04,0.96,0.95) and (1.2,1.08,1.14) for $k \in [1, 70]$, respectively.

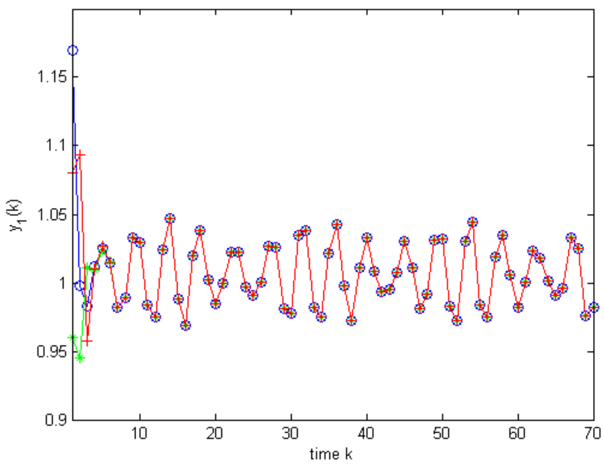

Figure2: Dynamic behavior of $y_1(k)$ of the solution $(x_1(k), y_1(k), y_2(k))$ to system (5.1) with the initial conditions (1.13,1.17,1.2), (1.04,0.96,0.95) and (1.2,1.08,1.14) for $k \in [1, 70]$, respectively.

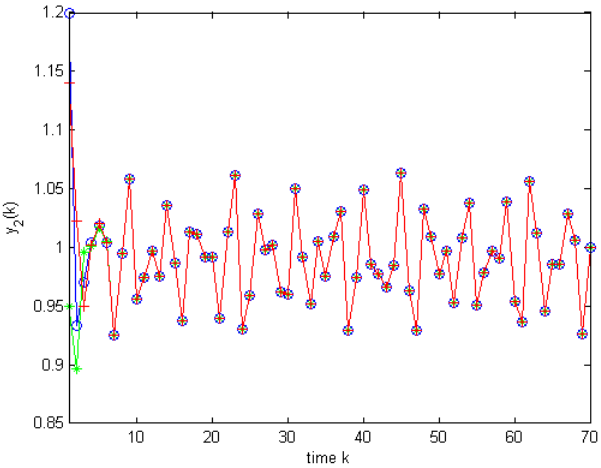

Figure3: Dynamic behavior of $y_2(k)$ of the solution $(x_1(k), y_1(k), y_2(k))$ to system (5.1) with the initial conditions (1.13,1.17,1.2), (1.04,0.96,0.95) and (1.2,1.08,1.14) for $k \in [1, 70]$, respectively.

6 Concluding Remarks

In Ref.[1], a discrete multispecies Lotka-Volterra competition predator-prey system is considered, in which the coefficients are all bounded non-negative sequence. Assuming that (H1)-(H3) and (3.1) hold, system (1.1) is globally attractive, which can be given in[1]. In this paper, assuming that the coefficients in system (1.1) are bounded non-negative almost periodic sequences, we obtain the sufficient conditions for the existence of a unique almost periodic solution which is globally attractive. By comparative analysis, we find that when the coefficients in system (1.1) are almost periodic, the existence of a unique almost periodic solution of system (1.1) is determined by the global attractivity of system (1.1), which implies that there is no additional condition to add.

Furthermore, for the almost periodic discrete multispecies Lotka-Volterra competition predator-prey system (1.1) with time delays or feedback controls, we would like to mention here the question of whether the existence of a unique almost periodic solution is determined by the global attractivity of the system or not. It is, in fact, a very challenging problem, and we leave it for our future work.

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Competing Interests

The authors declare that no competing interests exist.

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