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Integral representations for local dilogarithm and trilogarithm functions

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Abstract: We show new integral representations for dilogarithm and trilogarithm functions on the unit interval. As a consequence, we also prove (1) new integral representations for Apéry, Catalan constants and Legendre χ functions of order 2, 3, (2) a lower bound for the dilogarithm function on the unit interval, (3) new Euler sums.

Keywords: Apéry constant; Catalan constant; Dilogarithm; Euler sum; Inverse sine function; Riemann zeta function; Trilogarithm; Wallis integral.

MSC: Primary: 33E20; Secondary: 11G55; 11M06; 11M41.

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1. Introduction

Polylogarithm function

The *polylogarithm function*

$$\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z + \frac{z^2}{2^s} + \frac{z^3}{3^s} + \cdots, \quad s, z \in \mathbb{C}, |z| < 1$$

plays a significant role in many areas of number theory; its origin, the dilogarithm $\mathrm{Li}_2(z)$, dates back to Abel, Euler, Kummer, Landen and Spence etc. See Kirillov [1], Lewin [2], Zagier [3] for more details. The main theme of this article is to better understand the relation between the dilogarithm, trilogarithm $\mathrm{Li}_3(z)$ functions and zeta values $\zeta(2)$, $\zeta(3)$ (Apéry constant), $\zeta(4)$ in terms of new integral representations.

Main results

First, we wish to briefly explain work of Boo Rim Choe (1987) [4], Ewell (1990) [5] and Williams-Yue (1993) [6, p.1582-1583] which motivated us. Their common idea is that, from Maclaurin series involving $\sin^{-1} x$, they each derived certain infinite sums related to $\zeta(2)$ and $\zeta(3)$ with termwise Wallis integral. Figure 1 gives summary of this.

In this article, we reformulate their ideas introducing *Wallis operator* and naturally extend their results.

- We find new integral representations for $\mathrm{Li}_2(t)$, $\mathrm{Li}_3(t)$, Legendre χ functions of order 2, 3 and even for Apéry, Catalan constants (Theorems 2, 5, Corollaries 3, 6).

Table 1. Summary of Boo Rim Choe, Ewell and Williams-Yue's work

Boo Rim Choe	$\sin^{-1} x = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{x^{2n+1}}{2n+1}$	$\to$	$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{3}{4}\zeta(2) = \frac{\pi^2}{8}$
Ewell	$\frac{\sin^{-1} x}{x} = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{x^{2n}}{2n+1}$	$\to$	$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} = \frac{7}{8}\zeta(3)$
Williams-Yue	$\frac{(\sin^{-1} x)^2}{x} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(2n)!!}{(2n-1)!!} \frac{x^{2n-1}}{n^2}$	$\to$	$\frac{\pi}{8} \sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{\pi}{8}\zeta(3)$

- We give a lower bound for $\mathrm{Li}_2(t)$ on the unit interval (Theorem 7).
- Making use of $(\sin^{-1} x)^3$ and $(\sin^{-1} x)^4$, we prove new Euler sums (Theorem 8).

Notation

Throughout this paper, n denotes a nonnegative integer. Let

$$(2n)!! = 2n(2n-2)\cdots 4\cdot 2,$$
$$(2n-1)!! = (2n-1)(2n-3)\cdots 3\cdot 1.$$

In particular, we understand that $(-1)!! = 0!! = 1$. Moreover, let

$$w_n = \frac{(n-1)!!}{n!!}.$$

Notice the relation $w_{2n}w_{2n+1} = \frac{1}{2n+1}$ as we will see in the sequel.

Remark 1. 1. The sequence $\{w_n\}$ appears in Wallis integral as

$$\int_0^{\pi/2} \sin^n x = \begin{cases} \frac{\pi}{2} w_n & n \text{ even}, \\ w_n & n \text{ odd}. \end{cases}$$

2. It also appears in the literature in the disguise of central binomial coefficients as

$$w_{2n} = \frac{(2n-1)!!}{(2n)!!} = \frac{1}{2^{2n}}\binom{2n}{n}.$$

See Apéry [7], van der Poorten [8], for example.

Unless otherwise specified, t, u, x, y are real numbers. By $\sin^{-1} x$ and $\cos^{-1} x$, we mean the real inverse sine and cosine functions ($\arcsin x, \arccos x$), that is,

$$\begin{aligned} y = \sin^{-1} x &\iff x = \sin y, \quad -\tfrac{\pi}{2} \le y \le \tfrac{\pi}{2}, \\ y = \cos^{-1} x &\iff x = \cos y, \quad 0 \le y \le \pi. \end{aligned}$$

Fact 1 (Gradshteyn-Ryzhik [9, p.60, 61])**.**

$$\sin^{-1} t = \sum_{n=0}^{\infty} w_{2n} \frac{t^{2n+1}}{2n+1}, \quad |t| \le 1. \tag{1}$$

$$(\sin^{-1} t)^2 = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{w_{2n}} \frac{t^{2n}}{n^2}, \quad |t| \le 1. \tag{2}$$

Further, $\sinh^{-1} x = \log(x + \sqrt{x^2+1})$ $(x \in \mathbb{R})$ denotes the inverse hyperbolic sine function (some authors write $\operatorname{arsinh} x$, $\operatorname{arcsinh} x$ or $\operatorname{argsinh} x$ for this one).

2. Dilogarithm function

2.1. Definition

Definition 1. For $0 \le t \le 1$, the *dilogarithm function* is

$$\mathrm{Li}_2(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^2}.$$

In particular, $\mathrm{Li}_2(1) = \zeta(2) = \frac{\pi^2}{6}$.

It is possible to describe its *even part* by Li_2 itself since

$$\sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)^2} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{(t^2)^n}{n^2} = \frac{1}{4}\mathrm{Li}_2(t^2).$$

Its *odd part* is called the Legendre χ function of order 2:

$$\chi_2(t) = \sum_{n=1}^{\infty} \frac{t^{2n-1}}{(2n-1)^2}.$$

Here is a fundamental relation of these two parts.

Observation 1.

$$\mathrm{Li}_2(t) = \chi_2(t) + \frac{1}{4}\mathrm{Li}_2(t^2).$$

Definition 2. Define

$$\mathrm{Ti}_2(t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2} t^{2n-1}$$

as a signed analog of $\chi_2(t)$.

This is also called the *inverse tangent integral* of order 2 because of the integral representations

$$\mathrm{Ti}_2(t) = \int_0^t \frac{\tan^{-1} x}{x}\, dx.$$

2.2. Wallis operator

Let $\mathbb{R}[[t]]$ denote the set of power series in t over real coefficients. Set

$$F(t) = \{f \in \mathbb{R}[[t]] \ \ | f(t) \text{ is convergent for } |t| \le 1\}.$$

Definition 3. For $f \in F(t)$, define $W : F(t) \to F(t)$ by

$$Wf(t) = \int_0^1 f(tu) \frac{du}{\sqrt{1-u^2}}.$$

Call W the *Wallis operator*.

Remark 2. [9, p.17] Power series may be integrated and differentiated termwise inside the circle of convergence without changing the radius of convergence. In the sequel, we will frequently use this without mentioning explicitly.

It is now helpful to understand W coefficientwise.

Lemma 1. *Let $f(t) = \sum_{n=0}^{\infty} a_n t^n \in F(t)$. Then*

$$Wf(t) = \sum_{n=0}^{\infty} a_{2n} \left(\frac{\pi}{2} w_{2n}\right) t^{2n} + \sum_{n=0}^{\infty} a_{2n+1} w_{2n+1} t^{2n+1}.$$

Proof.

$$\begin{aligned}
Wf(t) &= \int_0^1 f(tu)\, \frac{du}{\sqrt{1-u^2}} \\
&= \int_0^1 \left(\sum_{n=0}^{\infty} a_{2n} t^{2n} u^{2n} + \sum_{n=0}^{\infty} a_{2n+1} t^{2n+1} u^{2n+1} \right) \frac{du}{\sqrt{1-u^2}} \\
&= \sum_{n=0}^{\infty} a_{2n} t^{2n} \int_0^1 u^{2n} \frac{du}{\sqrt{1-u^2}} + \sum_{n=0}^{\infty} a_{2n+1} t^{2n+1} \int_0^1 u^{2n+1} \frac{du}{\sqrt{1-u^2}}
\end{aligned}$$

$$= \sum_{n=0}^{\infty} a_{2n} \left(\frac{\pi}{2} w_{2n}\right) t^{2n} + \sum_{n=0}^{\infty} a_{2n+1} w_{2n+1} t^{2n+1}.$$

□

Observe that W is linear in the sense that $W(f+g) = W(f) + W(g)$ and $W(cf) = cW(f)$ for $f, g \in F(t), c \in \mathbb{R}$.

2.3. Main Theorem 1

Lemma 2. *All of the following are convergent power series for* $|t| \leq 1$.

$$\sin^{-1} t = \sum_{n=0}^{\infty} w_{2n} \frac{t^{2n+1}}{2n+1}. \tag{3}$$

$$\frac{1}{2}(\sin^{-1} t)^2 = \sum_{n=1}^{\infty} \frac{1}{w_{2n}} \frac{t^{2n}}{(2n)^2}. \tag{4}$$

$$\sin^{-1} t + \frac{1}{\pi}(\sin^{-1} t)^2 = \sum_{n=0}^{\infty} w_{2n} \frac{t^{2n+1}}{2n+1} + \sum_{n=1}^{\infty} \frac{2}{\pi w_{2n}} \frac{t^{2n}}{(2n)^2}. \tag{5}$$

$$\sinh^{-1} t = \sum_{n=0}^{\infty} (-1)^n w_{2n} \frac{t^{2n+1}}{2n+1}. \tag{6}$$

$$\frac{1}{2}(\sinh^{-1} t)^2 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{w_{2n}} \frac{t^{2n}}{(2n)^2}. \tag{7}$$

Proof. We already saw (3) and (4) in Introduction. (5) is (3) + $\frac{2}{\pi}$(4). (6) and (7) follow from (3), (4) and $\sinh^{-1} z = \frac{1}{i} \sin^{-1}(iz)$ (for all $z \in \mathbb{C}$) [9, p.56]. □

Theorem 2. *For* $0 \leq t \leq 1$, *all of the following hold;*

$$\chi_2(t) = \int_0^1 \frac{\sin^{-1}(tu)}{\sqrt{1-u^2}} du. \tag{8}$$

$$\frac{1}{4} Li_2(t^2) = \frac{1}{\pi} \int_0^1 \frac{(\sin^{-1}(tu))^2}{\sqrt{1-u^2}} du. \tag{9}$$

$$Li_2(t) = \int_0^1 \frac{\sin^{-1}(tu) + \frac{1}{\pi}(\sin^{-1}(tu))^2}{\sqrt{1-u^2}} du. \tag{10}$$

$$Ti_2(t) = \int_0^1 \frac{\sinh^{-1}(tu)}{\sqrt{1-u^2}} du. \tag{11}$$

$$\frac{\pi}{2}\left(\frac{1}{4} Li_2(t^2) - \frac{1}{8} Li_2(t^4)\right) = \int_0^1 \frac{\frac{1}{2}(\sinh^{-1} tu)^2}{\sqrt{1-u^2}} du. \tag{12}$$

Proof. Note that these are equivalent to the following statements:

$$W\left(\sin^{-1} t\right) = \chi_2(t). \tag{13}$$

$$W\left(\frac{1}{2}(\sin^{-1} t)^2\right) = \frac{\pi}{2} \cdot \frac{1}{4} \mathrm{Li}_2(t^2). \tag{14}$$

$$W\left(\sin^{-1} t + \frac{1}{\pi}(\sin^{-1} t)^2\right) = \mathrm{Li}_2(t). \tag{15}$$

$$W(\sinh^{-1} t) = \mathrm{Ti}_2(t). \tag{16}$$

$$W\left(\frac{1}{2}(\sinh^{-1} t)^2\right) = \frac{\pi}{2}\left(\frac{1}{4}\mathrm{Li}_2(t^2) - \frac{1}{8}\mathrm{Li}_2(t^4)\right). \tag{17}$$

With Lemmas 1 and 2, we can verify (13)-(16) by checking coefficients of those series. For example,

$$W(\sin^{-1} t) = W\left(\sum_{n=0}^{\infty} w_{2n} \frac{t^{2n+1}}{2n+1}\right) = \sum_{n=0}^{\infty} w_{2n} w_{2n+1} \frac{t^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)^2} = \chi_2(t).$$

It remains to show (17).

$$
\begin{aligned}
W\left(\frac{1}{2}(\sinh^{-1} t)^2\right) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{w_{2n}} \left(w_{2n}\frac{\pi}{2}\right) \frac{t^{2n}}{(2n)^2} \\
&= \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n)^2} t^{2n} \\
&= \frac{\pi}{2} \left(\sum_{n=1}^{\infty} \frac{1}{(2n)^2} t^{2n} - 2 \sum_{n=1}^{\infty} \frac{1}{(4n)^2} t^{4n} \right) \\
&= \frac{\pi}{2} \left(\frac{1}{4} \mathrm{Li}_2(t^2) - \frac{1}{8} \mathrm{Li}_2(t^4) \right).
\end{aligned}
$$

□

Corollary 3.

$$\int_0^1 \frac{\sin^{-1} u}{\sqrt{1-u^2}} du = \frac{3}{4}\zeta(2) = \frac{\pi^2}{8}. \tag{18}$$

$$\frac{2}{\pi}\int_0^1 \frac{\frac{1}{2}(\sin^{-1} u)^2}{\sqrt{1-u^2}} du = \frac{1}{4}\zeta(2) = \frac{\pi^2}{24}. \tag{19}$$

$$\int_0^1 \left(\sin^{-1} u + \frac{1}{\pi}(\sin^{-1} u)^2\right) \frac{du}{\sqrt{1-u^2}} = \zeta(2) = \frac{\pi^2}{6}. \tag{20}$$

$$\int_0^1 \frac{\sinh^{-1} u}{\sqrt{1-u^2}} du = G. \tag{21}$$

$$\int_0^1 \frac{\frac{1}{2}(\sinh^{-1} u)^2}{\sqrt{1-u^2}} du = \frac{\pi}{16}\zeta(2) = \frac{\pi^3}{96}. \tag{22}$$

Proof. These are $\chi_2(1), \frac{1}{4}\mathrm{Li}_2(1^2), \mathrm{Li}_2(1), \mathrm{Ti}_2(1)$ and $\frac{\pi}{2}\left(\frac{1}{4}\mathrm{Li}_2(1^2) - \frac{1}{8}\mathrm{Li}_2(1^4)\right)$. □

3. Trilogarithm function

3.1. Definition

Definition 4. The *trilogarithm function* for $0 \le t \le 1$ is

$$\mathrm{Li}_3(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^3}.$$

Its odd part is the Legendre χ function of order 3:

$$\chi_3(t) = \sum_{n=1}^{\infty} \frac{t^{2n-1}}{(2n-1)^3}.$$

In particular, $\mathrm{Li}_3(1) = \zeta(3)$ and $\chi_3(1) = \frac{7}{8}\zeta(3)$.

Observation 4.

$$\mathrm{Li}_3(t) = \chi_3(t) + \frac{1}{8}\mathrm{Li}_3(t^2).$$

Further, a signed analog of $\chi_3(t)$ is

$$\mathrm{Ti}_3(t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3} t^{2n-1}.$$

3.2. Main Theorem 2

Lemma 3.

$$\int_0^t \frac{\sin^{-1} y}{y}\,dy = \sum_{n=0}^{\infty} w_{2n} \frac{t^{2n+1}}{(2n+1)^2}. \tag{23}$$

$$\int_0^t \frac{\frac{1}{2}(\sin^{-1} y)^2}{y}\,dy = \sum_{n=1}^{\infty} \frac{1}{w_{2n}} \frac{t^{2n}}{(2n)^3}. \tag{24}$$

$$\int_0^t \frac{\sin^{-1} y + \frac{1}{\pi}(\sin^{-1} y)^2}{y}\,dy = \sum_{n=0}^{\infty} w_{2n} \frac{t^{2n+1}}{(2n+1)^2} + \sum_{n=1}^{\infty} \frac{2}{\pi w_{2n}} \frac{t^{2n}}{(2n)^3}. \tag{25}$$

$$\int_0^t \frac{\sinh^{-1} y}{y}\,dy = \sum_{n=0}^{\infty} (-1)^n w_{2n} \frac{t^{2n+1}}{(2n+1)^2}. \tag{26}$$

$$\int_0^t \frac{\frac{1}{2}(\sinh^{-1} y)^2}{y}\,dy = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{w_{2n}} \frac{t^{2n}}{(2n)^3}. \tag{27}$$

Proof. We can derive all of these by integrating (3)-(7) termwise. □

As a consequence, we obtain the equalities below (cf. (13)-(17)).

$$W\left(\int_0^t \frac{\sin^{-1} y}{y}\,dy\right) = \chi_3(t). \tag{28}$$

$$W\left(\int_0^t \frac{\frac{1}{2}(\sin^{-1} y)^2}{y}\,dy\right) = \frac{\pi}{2} \cdot \frac{1}{8}\mathrm{Li}_3(t^2). \tag{29}$$

$$W\left(\int_0^t \frac{\sin^{-1} y + \frac{1}{\pi}(\sin^{-1} y)^2}{y}\,dy\right) = \mathrm{Li}_3(t). \tag{30}$$

$$W\left(\int_0^t \frac{\sinh^{-1} y}{y}\,dy\right) = \mathrm{Ti}_3(t). \tag{31}$$

$$\begin{aligned} W\left(\int_0^t \frac{\frac{1}{2}(\sinh^{-1} y)^2}{y}\,dy\right) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{w_{2n}} \left(w_{2n}\frac{\pi}{2}\right) \frac{t^{2n}}{(2n)^3} \\ &= \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n)^3} t^{2n} \\ &= \frac{\pi}{2} \left(\sum_{n=1}^{\infty} \frac{1}{(2n)^3} t^{2n} - 2\sum_{n=1}^{\infty} \frac{1}{(4n)^3} t^{4n} \right) \\ &= \frac{\pi}{2} \left(\frac{1}{8}\mathrm{Li}_3(t^2) - \frac{1}{32}\mathrm{Li}_3(t^4) \right). \end{aligned} \tag{32}$$

In this way, the five functions above come to possess *double* integral representations. For example,

$$\chi_3(t) = \int_0^1 \left(\int_0^{tu} \frac{\sin^{-1} y}{y}\,dy \right) \frac{du}{\sqrt{1-u^2}}.$$

We can indeed simplify such integrals to *single* ones by exchanging order of integrals.

Theorem 5.

$$\chi_3(t) = \int_0^1 \frac{\sin^{-1}(tx)\cos^{-1} x}{x}\,dx. \tag{33}$$

$$\frac{1}{8}Li_3(t^2) = \frac{2}{\pi} \int_0^1 \frac{\frac{1}{2}(\sin^{-1}(tx))^2 \cos^{-1} x}{x}\,dx. \tag{34}$$

$$Li_3(t) = \int_0^1 \frac{\left(\sin^{-1}(tx) + \frac{1}{\pi}(\sin^{-1}(tx))^2\right)\cos^{-1} x}{x}\,dx. \tag{35}$$

$$Ti_3(t) = \int_0^1 \frac{\sinh^{-1}(tx)\cos^{-1}x}{x}dx. \tag{36}$$

$$\frac{\pi}{2}\left(\frac{1}{8}Li_3(t^2) - \frac{1}{32}Li_3(t^4)\right) = \int_0^1 \frac{\frac{1}{2}(\sinh^{-1}(tx))^2\cos^{-1}x}{x}dx. \tag{37}$$

Proof. We give a proof altogether. For $t = 0$, all the equalities hold as $0 = 0$. Suppose $0 < t \le 1$. Let

$$f(y) \in \left\{\sin^{-1}y, \frac{1}{\pi}(\sin^{-1}y)^2, \sin^{-1}y + \frac{1}{\pi}(\sin^{-1}y)^2, \sinh^{-1}y, \frac{1}{2}(\sinh^{-1}y)^2\right\}.$$

Then

$$\begin{aligned} W\left(\int_0^t \frac{f(y)}{y}\,dy\right) &= \int_0^1 \int_0^{tu} \frac{f(y)}{y}\,dy\,\frac{du}{\sqrt{1-u^2}} \\ &= \int_0^t \int_{y/t}^1 \frac{f(y)}{y}\frac{1}{\sqrt{1-u^2}}\,dudy \\ &= \int_0^t \frac{f(y)}{y}\cos^{-1}\frac{y}{t}\,dy \\ &= \int_0^1 \frac{f(tx)}{x}\cos^{-1}x\,dx. \end{aligned}$$

□

Corollary 6.

$$\int_0^1 \frac{\sin^{-1}x\cos^{-1}x}{x}dx = \frac{7}{8}\zeta(3). \tag{38}$$

$$\frac{2}{\pi}\int_0^1 \frac{\frac{1}{2}(\sin^{-1}x)^2\cos^{-1}x}{x}dx = \frac{1}{8}\zeta(3). \tag{39}$$

$$\int_0^1 \frac{\left(\sin^{-1}x + \frac{1}{\pi}(\sin^{-1}x)^2\right)\cos^{-1}x}{x}dx = \zeta(3). \tag{40}$$

$$\int_0^1 \frac{\sinh^{-1}x\cos^{-1}x}{x}dx = \frac{\pi^3}{32}. \tag{41}$$

$$\int_0^1 \frac{\frac{1}{2}(\sinh^{-1}x)^2\cos^{-1}x}{x}dx = \frac{3\pi}{64}\zeta(3). \tag{42}$$

Proof. These are $\chi_3(1), \frac{1}{8}\mathrm{Li}_3(1^2), \mathrm{Li}_3(1), \mathrm{Ti}_3(1)$ and $\frac{\pi}{2}\left(\frac{1}{8}\mathrm{Li}_3(1^2) - \frac{1}{32}\mathrm{Li}_3(1^4)\right)$. □

4. Applications

4.1. Inequalities

It is easy to see from the definitions $\mathrm{Li}_2(t) = \sum_{n=1}^{\infty}\frac{t^n}{n^2}$ and $\chi_2(t) = \sum_{n=1}^{\infty}\frac{t^{2n-1}}{(2n-1)^2}$ $(0 \le t \le 1)$ that

$$0 \le \mathrm{Li}_2(t) \le \frac{\pi^2}{6} \quad \text{and} \quad 0 \le \chi_2(t) \le \frac{\pi^2}{8}.$$

In fact, we can improve these inequalities a little more. For upper bounds, it is immediate that

$$\begin{aligned} \mathrm{Li}_2(t) &= \sum_{n=1}^{\infty}\frac{t^n}{n^2} \le \sum_{n=1}^{\infty}\frac{t}{n^2} = \frac{\pi^2}{6}t, \\ \chi_2(t) &= \sum_{n=1}^{\infty}\frac{t^{2n-1}}{(2n-1)^2} \le \sum_{n=1}^{\infty}\frac{t}{(2n-1)^2} = \frac{\pi^2}{8}t. \end{aligned}$$

We next prove nontrivial lower bounds for these functions and also $\mathrm{Ti}_2(t)$.

Theorem 7. *For $0 \le t \le 1$,*

$$Li_2(t) \ge \frac{4}{3\pi}\frac{(\sin^{-1} t)^3}{t}. \tag{43}$$

$$\chi_2(t) \ge \frac{(\sin^{-1} t)^2}{2t}. \tag{44}$$

$$Ti_2(t) \ge \frac{(\sinh^{-1} t)^2}{2t}. \tag{45}$$

Before the proof, we need a lemma. It provides another integral representation of $\mathrm{Li}_2(t)$ which seems interesting itself.

Lemma 4. *For $0 \le t \le 1$,*

$$Li_2(t) = \frac{8\sqrt{t}}{\pi}\int_0^1 \frac{\sin^{-1}(\sqrt{t}x)\cos^{-1} x}{\sqrt{1-tx^2}}\,dx. \tag{46}$$

$\left(cf.\quad Li_3(t) = \frac{8}{\pi}\int_0^1 \frac{(\sin^{-1}(\sqrt{t}x))^2\cos^{-1}x}{x}dx, \quad t \mapsto \sqrt{t} \text{ in } (34).\right)$

Proof. If $t = 0$, then both sides are 0. For $0 < t \le 1$,

$$\begin{aligned}
\text{RHS} &= \frac{8}{\pi}\int_0^{\sqrt{t}} \frac{\sin^{-1} y}{\sqrt{1-y^2}}\cos^{-1}\frac{y}{\sqrt{t}}\,dy \\
&= \frac{8}{\pi}\int_0^{\sqrt{t}} \frac{\sin^{-1} y}{\sqrt{1-y^2}}\int_{y/\sqrt{t}}^1 \frac{1}{\sqrt{1-u^2}}\,dudy \\
&= \frac{8}{\pi}\int_0^1\int_0^{\sqrt{tu}} \frac{\sin^{-1} y}{\sqrt{1-y^2}}dy\frac{1}{\sqrt{1-u^2}}\,du \\
&= \frac{8}{\pi}W\left(\int_0^{\sqrt{t}} \frac{\sin^{-1} y}{\sqrt{1-y^2}}dy\right) \\
&= \frac{8}{\pi}W\left(\frac{1}{2}\left(\sin^{-1}\sqrt{t}\right)^2\right) \\
&= \frac{8}{\pi}\left(\frac{\pi}{2}\frac{1}{4}\mathrm{Li}_2\left((\sqrt{t})^2\right)\right) = \mathrm{Li}_2(t).
\end{aligned}$$

□

Proof of Theorem 7. If $t = 0$, then all of (43)-(45) hold as $0 \ge 0$. Suppose $t > 0$. Since $\sin^{-1}$ is increasing on $[0,1]$, $\sin^{-1}(tx) \le \sin^{-1}(\sqrt{t}x)$ for all $0 < t, x \le 1$. Then

$$\begin{aligned}
\mathrm{Li}_2(t) &= \frac{8\sqrt{t}}{\pi}\int_0^1 \frac{\sin^{-1}(\sqrt{t}x)}{\sqrt{1-tx^2}}\cos^{-1} x\,dx \\
&\ge \frac{8t}{\pi}\int_0^1 \frac{\sin^{-1}(tx)}{\sqrt{1-t^2x^2}}\cos^{-1} x\,dx \\
&= \frac{8t}{\pi}\int_0^1 \frac{1}{t}\left(\frac{1}{2}(\sin^{-1} tx)^2\right)\cos^{-1} x\,dx \\
&= \frac{8}{\pi}\left(\underbrace{\left[\frac{1}{2}(\sin^{-1} tx)^2\cos^{-1} x\right]_0^1}_{0} - \int_0^1 \frac{1}{2}(\sin^{-1} tx)^2\frac{-1}{\sqrt{1-x^2}}\,dx\right) \\
&\ge \frac{4}{\pi}\int_0^1 \frac{(\sin^{-1} tx)^2}{\sqrt{1-t^2x^2}}\,dx \\
&= \frac{4}{\pi}\left[\frac{1}{3t}(\sin^{-1} tx)^3\right]_0^1 = \frac{4}{3\pi}\frac{(\sin^{-1} t)^3}{t}.
\end{aligned}$$

Next, we prove (44). Note that

$$\frac{\sin^{-1}(tx)}{\sqrt{1-t^2x^2}} \le \frac{\sin^{-1}(tx)}{\sqrt{1-x^2}}$$

for $0 < t, x < 1$. Integrate these from 0 to 1 in x so that

$$\int_0^1 \frac{\sin^{-1}(tx)}{\sqrt{1-t^2x^2}}\,dx \le \int_0^1 \frac{\sin^{-1}(tx)}{\sqrt{1-x^2}}\,dx,$$
$$\left[\frac{(\sin^{-1}(tx))^2}{2t}\right]_0^1 \le \chi_2(t),$$
$$\frac{(\sin^{-1}t)^2}{2t} \le \chi_2(t).$$

Quite similarly, for $0 < t, x < 1$, it also holds that

$$\frac{\sinh^{-1}(tx)}{\sqrt{1+t^2x^2}} \le \frac{\sinh^{-1}(tx)}{\sqrt{1-x^2}},$$
$$\int_0^1 \frac{\sinh^{-1}(tx)}{\sqrt{1+t^2x^2}}\,dx \le \int_0^1 \frac{\sinh^{-1}(tx)}{\sqrt{1-x^2}}\,dx = \mathrm{Ti}_2(t).$$

The left hand side is

$$\left[\frac{\sinh^{-1}(tx)^2}{2t}\right]_0^1 = \frac{(\sinh^{-1}t)^2}{2t}.$$

□

4.2. Euler sums

Definition 5. A *harmonic number* is $H_n = \sum_{k=1}^{n} \frac{1}{k}$. More generally, for $m, n \ge 1$, an (m,n)-*harmonic number* is

$$H_n^{(m)} = \sum_{k=1}^{n} \frac{1}{k^m}.$$

In particular, $H_n^{(1)} = H_n$. Any series involving such numbers is called an *Euler sum*.
Vălean [10, p.292-293] presents truly remarkable Euler sums such as

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^2} = \frac{17}{4}\zeta(4),$$
$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^3} = \frac{7}{2}\zeta(5) - \zeta(2)\zeta(3),$$
$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^4} = \frac{97}{24}\zeta(6) - 2\zeta^2(3),$$
$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^5} = 6\zeta(7) - \zeta(2)\zeta(5) - \frac{5}{2}\zeta(3)\zeta(4),$$
$$\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^2} = \zeta(2)\zeta(3) + \zeta(5).$$

There are many ideas to prove such formulas; Borwein and Bradley [11] gives thirty two proofs for

$$\sum_{n=1}^{\infty} \frac{H_{n-1}}{n^2} = \zeta(3) = 8\sum_{n=1}^{\infty} \frac{(-1)^n H_{n-1}}{n^2}$$

by integrals, polylogarithm functions, Fourier series and hypergeometric functions etc. Here, as an application of our main idea, Wallis operators, we prove two new Euler sums. Let

$$O_n^{(2)} = H_{2n-1}^{(2)} - \frac{1}{4}H_{n-1}^{(2)} = \sum_{k=1}^{2n-1}\frac{1}{k^2} - \sum_{k=1}^{n-1}\frac{1}{(2k)^2} = \sum_{k=0}^{n-1}\frac{1}{(2k+1)^2}.$$

Theorem 8.

$$\sum_{n=0}^{\infty}\frac{O_n^{(2)}}{(2n+1)^2} = \frac{\pi^4}{384} = \frac{15}{64}\zeta(4). \tag{47}$$

$$\sum_{n=1}^{\infty}\frac{H_{n-1}^{(2)}}{n^2} = \frac{\pi^4}{120} = \frac{3}{4}\zeta(4). \tag{48}$$

For the proof, we make use of less-known Maclaurin series for $(\sin^{-1} t)^3$ and $(\sin^{-1} t)^4$; thus we can interpret this result as a natural subsequence of Boo, Ewell and Williams-Yue's work.

Lemma 5.

$$(\sin^{-1} t)^3 = \sum_{n=0}^{\infty}\left(6O_n^{(2)}\right) w_{2n}\frac{t^{2n+1}}{2n+1}. \tag{49}$$

$$(\sin^{-1} t)^4 = \frac{1}{2}\sum_{n=1}^{\infty}\left(3H_{n-1}^{(2)}\right)\frac{1}{w_{2n}}\frac{t^{2n}}{n^2}. \tag{50}$$

$$\left(\textit{cf.}\quad \sin^{-1} t = \sum_{n=0}^{\infty} w_{2n}\frac{t^{2n+1}}{2n+1}, \quad (\sin^{-1} t)^2 = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{w_{2n}}\frac{t^{2n}}{n^2}\right).$$

Proof. First, write $(\sin^{-1} t)^3 = \sum_{n=0}^{\infty} A_n t^{2n+1}$, $A_n \in \mathbb{R}$ and let $a_n = \frac{2n+1}{w_{2n}}A_n$ $(n \geq 0)$. It is enough to show that $a_n = 6O_n^{(2)}$. Since the series $(\sin^{-1} t)^3 = (t + \frac{t^3}{6} + \cdots)^3$ starts from the t^3 term, $A_0 = a_0 = 0$. For convenience, set

$$f_n(x) = \frac{\sin^{2n+1} x}{(2n+1)!}.$$

Then

$$f_n'(x) = \frac{\sin^{2n} x}{(2n)!}\cos x,$$

$$f_n''(x) = \frac{1}{(2n)!}\left(2n\sin^{2n-1} x(1-\sin^2 x) - \sin^{2n+1} x\right) = f_{n-1}(x) - (2n+1)^2 f_n(x).$$

Now let $x = \sin^{-1} t$ $(-\frac{\pi}{2} \leq x \leq \frac{\pi}{2})$, $b_n = (2n-1)!!$. Recall that

$$\sin^{-1} t = \sum_{n=0}^{\infty} w_{2n}\frac{t^{2n+1}}{2n+1}.$$

In terms of x, b_n, $f_n(x)$, this is

$$x = \sum_{n=0}^{\infty} w_{2n}\frac{\sin^{2n+1} x}{2n+1} = \sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!}(2n)!\frac{\sin^{2n+1} x}{(2n+1)!} = \sum_{n=0}^{\infty} b_n^2 f_n(x).$$

Thus,

$$x^3 = \sum_{n=0}^{\infty} A_n \sin^{2n+1} x = \sum_{n=0}^{\infty} a_n\left(w_{2n}\frac{\sin^{2n+1} x}{2n+1}\right) = \sum_{n=0}^{\infty} a_n b_n^2 f_n(x).$$

Differentiate both sides twice in x:

$$\begin{aligned}
6x &= \sum_{n=0}^{\infty} a_n b_n^2 f_n''(x)\\
&= \sum_{n=0}^{\infty} a_n b_n^2 (f_{n-1}(x) - (2n+1)^2 f_n(x))\\
&= \sum_{n=0}^{\infty} (a_{n+1} b_{n+1}^2 f_n(x) - a_n b_n^2 (2n+1)^2 f_n(x)),\\
6\sum_{n=0}^{\infty} b_n^2 f_n(x) &= \sum_{n=0}^{\infty} (a_{n+1} b_{n+1}^2 f_n(x) - a_n b_n^2 (2n+1)^2 f_n(x)).
\end{aligned}$$

Equating coefficients of $f_n(x)$ yields

$$6b_n^2 = a_{n+1} b_{n+1}^2 - a_n b_n^2 (2n+1)^2, \quad n \geq 0.$$

Since $b_{n+1} = (2n+1)b_n$ and $b_n \neq 0$, we must have

$$a_{n+1} - a_n = \frac{6}{(2n+1)^2}.$$

With $a_0 = 0$, we now arrive at

$$a_n = \sum_{k=0}^{n-1} \frac{6}{(2k+1)^2} = 6O_n^{(2)},$$

as required.

The proof for (50) proceeds along the same line. Write $(\sin^{-1} t)^4 = \frac{1}{2}\sum_{n=0}^{\infty} C_n t^{2n}$, $C_n \in \mathbb{R}$ and let $c_n = C_n w_{2n} n^2$ $(n \geq 1)$. It is enough to show that $c_n = 3H_{n-1}^{(2)}$. Since the series $(\sin^{-1} t)^4$ starts from the t^4 term, $C_1 = c_1 = 0$. For convenience, set

$$g_n(x) = \frac{\sin^{2n} x}{(2n)!}.$$

Then

$$g_n'(x) = \frac{\sin^{2n-1} x}{(2n-1)!}\cos x,$$

$$g_n''(x) = \frac{1}{(2n-1)!}\left((2n-1)\sin^{2n-2} x(1-\sin^2 x) - \sin^{2n} x\right) = g_{n-1}(x) - (2n)^2 g_n(x).$$

Now let $x = \sin^{-1} t$ $(-\frac{\pi}{2} \leq x \leq \frac{\pi}{2})$, $d_n = 2^n (n-1)!$. Recall that

$$(\sin^{-1} t)^2 = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{w_{2n}} \frac{t^{2n}}{n^2}.$$

In terms of $x, d_n, g_n(x)$, this is

$$x^2 = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{w_{2n}} \frac{\sin^{2n} x}{n^2} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{(2n)!!}{(2n-1)!!} \frac{(2n)!}{n^2} \frac{\sin^{2n} x}{(2n)!} = \frac{1}{2}\sum_{n=1}^{\infty} d_n^2 g_n(x).$$

Thus,

$$x^4 = \frac{1}{2}\sum_{n=1}^{\infty} c_n \left(\frac{1}{w_{2n}} \frac{\sin^{2n} x}{n^2}\right) = \frac{1}{2}\sum_{n=1}^{\infty} c_n d_n^2 g_n(x).$$

Differentiate both sides twice in x:

$$\begin{aligned}
12x^2 &= \frac{1}{2}\sum_{n=1}^{\infty} c_n d_n^2 g_n''(x)\\
&= \frac{1}{2}\sum_{n=1}^{\infty} c_n d_n^2 (g_{n-1}(x) - (2n)^2 g_n(x))
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{n=0}^{\infty} c_{n+1}d_{n+1}^2 g_n(x) - \frac{1}{2}\sum_{n=1}^{\infty} c_n d_n^2 (2n)^2 g_n(x) \\
&= \frac{1}{2}\sum_{n=1}^{\infty} c_{n+1}d_{n+1}^2 g_n(x) - \frac{1}{2}\sum_{n=1}^{\infty} c_n d_n^2 (2n)^2 g_n(x) \quad (c_1 = 0) \\
&= \frac{1}{2}\sum_{n=1}^{\infty} (c_{n+1}d_{n+1}^2 - c_n d_n^2 (2n)^2) g_n(x), \\
\frac{12}{2}\sum_{n=1}^{\infty} d_n^2 g_n(x) &= \frac{1}{2}\sum_{n=1}^{\infty} (c_{n+1}d_{n+1}^2 - c_n d_n^2 (2n)^2) g_n(x).
\end{aligned}
$$

Equating coefficients of $g_n(x)$ yields

$$\frac{12}{2}d_n^2 = \frac{1}{2}(c_{n+1}d_{n+1}^2 - c_n d_n^2(2n)^2), \quad n \geq 1.$$

Since $d_{n+1} = 2nd_n$ and $d_n \neq 0$, we must have

$$c_{n+1} - c_n = \frac{12}{(2n)^2}.$$

With $c_1 = 0$, we conclude that

$$c_n = \sum_{k=1}^{n-1} \frac{12}{(2k)^2} = \sum_{k=1}^{n-1} \frac{3}{k^2} = 3H_{n-1}^{(2)}.$$

□

Proof of Theorem 8. Note that

$$
\begin{aligned}
W\left(\frac{1}{6}(\sin^{-1} t)^3\right) &= \sum_{n=1}^{\infty} \frac{\widetilde{O}_n^{(2)} w_{2n}}{2n+1} w_{2n+1} t^{2n+1} \\
&= \sum_{n=1}^{\infty} \frac{\widetilde{O}_n^{(2)}}{(2n+1)^2} t^{2n+1}.
\end{aligned}
$$

Clearly, $t = 1$ gives the sum for (47). Therefore,

$$W\left(\frac{1}{6}(\sin^{-1} t)^3\right)\bigg|_{t=1} = \int_0^1 \frac{1}{6}(\sin^{-1} u)^3 \frac{du}{\sqrt{1-u^2}} = \left[\frac{1}{24}(\sin^{-1} u)^4\right]_0^1 = \frac{\pi^4}{384}.$$

Similarly, we have

$$W\left(\frac{2}{3}(\sin^{-1} t)^4\right) = \frac{\pi}{2}\sum_{n=1}^{\infty} \frac{H_{n-1}^{(2)}}{n^2} t^{2n}$$

so that

$$W\left(\frac{2}{3}(\sin^{-1} t)^4\right)\bigg|_{t=1} = \int_0^1 \frac{2}{3}(\sin^{-1} u)^4 \frac{du}{\sqrt{1-u^2}} = \left[\frac{2}{15}(\sin^{-1} u)^5\right]_0^1 = \frac{\pi^5}{240}.$$

We conclude that

$$\sum_{n=1}^{\infty} \frac{H_{n-1}^{(2)}}{n^2} = \frac{2}{\pi}\left(\frac{\pi^5}{240}\right) = \frac{\pi^4}{120}.$$

□

Remark 3. 1. (47) is a variation of De Doelder's formula $\sum_{n=1}^{\infty} \frac{O_n^{(2)}}{n^2} = \frac{\pi^4}{32}$ [12, p.1196 (13)] and (48) gives another proof of $\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2} = \frac{7}{4}\zeta(4)$ [10, p.286] because

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2} = \sum_{n=1}^{\infty} \left(\frac{H_{n-1}^{(2)}}{n^2} + \frac{1}{n^4}\right) = \frac{3}{4}\zeta(4) + \zeta(4) = \frac{7}{4}\zeta(4).$$

2. After preparation of the manuscript, Christophie Vignat kindly told me that recently Guo-Lim-Qi (2021) [13] described Maclaurin series of integer powers of arcsin. In fact, it was the result from J.M. Borwein-Chamberland (2007) [14].

4.3. Integral evaluation

As byproduct of our discussions, we find evaluation of many integrals with known special values of $\mathrm{Li}_2(t), \mathrm{Li}_3(t)$. Here, we record several examples. Let $\phi = \frac{1+\sqrt{5}}{2}$ be the golden ratio. Observe that

$$\phi^{-1} = \frac{\sqrt{5}-1}{2}, \quad \phi^{-2} = \frac{3-\sqrt{5}}{2}.$$

We write $\log^2 x$ for $(\log x)^2$. Note that

$$\log^2(\phi^{-1}) = (\log(\phi^{-1}))^2 = (-\log(\phi))^2 = (\log(\phi))^2 = \log^2(\phi).$$

Fact 2 ([2]).

$$\mathrm{Li}_2(\phi^{-1}) = -\log^2(\phi) + \frac{\pi^2}{10}. \tag{51}$$

$$\mathrm{Li}_2(\phi^{-2}) = -\log^2(\phi) + \frac{\pi^2}{15}. \tag{52}$$

$$\mathrm{Li}_3(\phi^{-2}) = \frac{4}{5}\zeta(3) - \frac{2\pi^2}{15}\log\phi + \frac{2}{3}\log^3\phi. \tag{53}$$

$$\mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{1}{2}\log^2 2. \tag{54}$$

$$\mathrm{Li}_3\left(\frac{1}{2}\right) = \frac{7}{8}\zeta(3) - \frac{\pi^2}{12}\log 2 + \frac{1}{6}\log^3 2. \tag{55}$$

Corollary 9.

$$\int_0^1 \frac{\sin^{-1}(\phi^{-1}u)}{\sqrt{1-u^2}}\,du = -\frac{3}{4}\log^2(\phi) + \frac{\pi^2}{12}. \tag{56}$$

$$\int_0^1 \frac{\frac{1}{2}(\sin^{-1}\frac{u}{\sqrt{2}})^2}{\sqrt{1-u^2}}\,du = \frac{\pi}{8}\left(\frac{\pi^2}{12} - \frac{1}{2}\log^2 2\right). \tag{57}$$

$$\frac{16}{\pi}\int_0^1 \frac{1}{2}\left(\sin^{-1}\phi^{-1}x\right)^2 \frac{\cos^{-1}x}{x}\,dx = \frac{4}{5}\zeta(3) - \frac{2\pi^2}{15}\log\phi + \frac{2}{3}\log^3\phi. \tag{58}$$

$$\int_0^1 \frac{1}{2}\left(\sinh^{-1}\phi^{-1/2}u\right)^2 \frac{du}{\sqrt{1-u^2}} = \frac{\pi}{2}\left(-\frac{1}{8}\log^2\phi + \frac{\pi^2}{60}\right). \tag{59}$$

$$\frac{16}{\pi}\int_0^1 \frac{1}{2}\left(\sin^{-1}\frac{x}{\sqrt{2}}\right)^2 \frac{\cos^{-1}x}{x}\,dx = \frac{7}{8}\zeta(3) - \frac{\pi^2}{12}\log 2 + \frac{1}{6}\log^3 2. \tag{60}$$

Proof.

$$\begin{aligned}\int_0^1 \frac{\sin^{-1}(\phi^{-1}u)}{\sqrt{1-u^2}}\,du &= \chi_2(\phi^{-1}) = \mathrm{Li}_2(\phi^{-1}) - \frac{1}{4}\mathrm{Li}_2(\phi^{-2}) \\ &= \left(-\log^2(\phi) + \frac{\pi^2}{10}\right) - \frac{1}{4}\left(-\log^2(\phi) + \frac{\pi^2}{15}\right) = -\frac{3}{4}\log^2(\phi) + \frac{\pi^2}{12}.\end{aligned}$$

$$W\left(\frac{1}{2}(\sin^{-1}t)^2\right)\Big|_{t=1/\sqrt{2}} = \frac{\pi}{8}\mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{\pi}{8}\left(\frac{\pi^2}{12} - \frac{1}{2}\log^2 2\right).$$

(34) for $t = \phi^{-1}$ with (53) gives (58).

$$W(\frac{1}{2}(\sinh^{-1}t)^2)|_{t=\phi^{-1/2}} = \frac{\pi}{2}\left(\frac{1}{4}\mathrm{Li}_2(\phi^{-1}) - \frac{1}{8}\mathrm{Li}_2(\phi^{-2})\right) = \frac{\pi}{2}\left(-\frac{1}{8}\log^2\phi + \frac{\pi^2}{60}\right).$$

Finally, (34) for $t = 1/\sqrt{2}$ with (55) gives (60). □

5. Concluding remarks

Here, we record several remarks for our future research.

1. For $0 \le \alpha \le 1$, define a generalized Wallis operator

$$W_\alpha f(t) = \int_0^\alpha f(tu) \frac{du}{\sqrt{1-u^2}}$$

so that we can deal with more general integrals. Study W_α, particularly for $\alpha = 1/2, \sqrt{2}/2, \sqrt{3}/2$.
2. Can we show any inequality for $\mathrm{Li}_3(t), \chi_3(t)$ and $\mathrm{Ti}_3(t)$ in a similar way?
3. Discuss $(\sinh^{-1} t)^3$, $(\sinh^{-1} t)^4$ and related Euler sums.
4. Wolfram alpha [15] says that

$$\int_0^1 \frac{(\sin^{-1} x)^3}{x} dx = \int_0^{\pi/2} u^3 \cot u \, du = \frac{\pi^3}{8}\log 2 - \frac{9}{16}\pi\zeta(3),$$
$$\int_0^1 \frac{(\sin^{-1} x)^4}{x} dx = \int_0^{\pi/2} u^4 \cot u \, du = \frac{1}{32}\left(-18\pi^2\zeta(3) + 93\zeta(5) + 2\pi^4 \log 2\right).$$

It should be possible to describe such integrals as certain infinite sums with or without numbers w_{2n}. We plan to study those details in subsequent publication.
5. It is interesting that (38) happens to be quite similar to

$$\int_0^1 \frac{\tan^{-1} x \cot^{-1} x}{x} dx = \frac{7}{8}\zeta(3).$$

Not often this result appears in this form in the literature, though. Now, let us see how we evaluate this integral. Let

$$I = \int_0^1 \frac{\tan^{-1} x \cot^{-1} x}{x} dx,$$
$$I_1 = \int_0^1 \frac{\tan^{-1} x}{x} dx,$$
$$I_2 = \int_0^1 \frac{(\tan^{-1} x)^2}{x} dx.$$

Then

$$I = \int_0^1 \frac{\tan^{-1} x \cot^{-1} x}{x} dx$$
$$= \int_0^1 \frac{\tan^{-1} x \left(\frac{\pi}{2} - \tan^{-1} x\right)}{x} dx = \frac{\pi}{2} I_1 - I_2.$$

We can compute I_1 and I_2 as follows.

$$I_1 = \int_0^1 \frac{\tan^{-1} x}{x} dx = \int_0^1 \sum_{n=0}^\infty \frac{(-1)^n}{2n+1} x^{2n} dx$$
$$= \sum_{n=0}^\infty \frac{(-1)^n}{2n+1} \int_0^1 x^{2n} dx = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^2} = G.$$

For I_2, recall from Fourier analysis that

$$\log\left(\tan\frac{y}{2}\right) = -2\sum_{n=0}^\infty \frac{1}{2n+1}\cos(2n+1)y, \quad 0 < y < \pi.$$

It follows that

$$\begin{aligned}
I_2 &= \int_0^1 \frac{(\tan^{-1} x)^2}{x}\, dx =_{[y=2\tan^{-1} x]} \frac{1}{4}\int_0^{\pi/2} \frac{y^2}{\sin y}\, dy \\
&= \frac{1}{4}\left(\left[y^2 \log\left(\tan\frac{y}{2}\right)\right]_0^{\pi/2} - \int_0^{\pi/2} 2y\log\left(\tan\frac{y}{2}\right) dy\right) \\
&= -\frac{1}{2}\int_0^{\pi/2} y\left(-2\sum_{n=0}^{\infty}\frac{1}{2n+1}\cos(2n+1)y\right) dy \\
&= \sum_{n=0}^{\infty}\frac{1}{2n+1}\int_0^{\pi/2} y\cos(2n+1)y\, dy \\
&= \sum_{n=0}^{\infty}\frac{1}{2n+1}\left(\left[y\frac{\sin(2n+1)y}{2n+1}\right]_0^{\pi/2} - \int_0^{\pi/2}\frac{\sin(2n+1)y}{2n+1}\, dy\right) \\
&= \sum_{n=0}^{\infty}\left(\frac{\pi}{2}\frac{(-1)^n}{(2n+1)^2} - \frac{1}{(2n+1)^3}\right) = \frac{\pi G}{2} - \frac{7}{8}\zeta(3).
\end{aligned}$$

Finally, we see

$$I = \frac{\pi G}{2} - \left(\frac{\pi G}{2} - \frac{7}{8}\zeta(3)\right) = \frac{7}{8}\zeta(3).$$

Open Question What if we replace $\tan^{-1}$ by $\tanh^{-1}$?

In this article, we encountered many integral representations for dilogarithm, trilogarithm and hence $\zeta(2)$, the Catalan constant G and $\zeta(3)$ as a reformulation of Boo Rim Choe (1987) [4], Ewell (1990) [5] and Williams-Yue (1993) [6] on the inverse sine function. As an application, we also proved new Euler sums. Indeed, there are subsequent results on multiple zeta and t-values $\zeta(3,2,\cdots,2)$, $t(3,2,\cdots,2)$ as Hoffman and Zagier discussed in [16,17]. We will write them with more details at another opportunity.

Conflicts of Interest: "The author declares no conflict of interest".

References

[1] Kirillov, A. N. (1995). Dilogarithm identities. *Progress of Theoretical Physics Supplement, 118*, 61-142.

[2] Lewin, L. (1981). *Polylogarithms and Associated Functions*. North-Holland, Amsterdam.

[3] Zagier, D. (2007). The dilogarithm function. *In Frontiers in number theory, physics, and geometry II* (pp. 3-65). Springer, Berlin, Heidelberg.

[4] Choe, B. R. (1987). An elementary proof of $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$. *The American Mathematical Monthly, 94*(7), 662-663.

[5] Ewell, J. A. (1990). A new series representation for $\zeta(3)$. *The American Mathematical Monthly, 97*(3), 219-220.

[6] Yue, Z. N., & Williams, K. S. (1993). Some series representations of $\zeta(2n+1)$. *The Rocky Mountain Journal of Mathematics, 23*(4), 1581-1592.

[7] Apéry, R. (1979). Irrationalité de $\zeta(2)$ et $\zeta(3)$. *Astérisque, 61*, 11-13.

[8] van der Poorten, A. (2005). A proof that Euler missed, Apéry's proof of the irrationality of $\zeta(3)$, an informal report. *ALF's reprints, Paper, 45*, 1-16.

[9] Gradshteyn, I. S., & Ryzhik, I. M. (2014). *Table of Integrals, Series, and Products*. Academic press.

[10] Vălean, C. I. (2019). *(Almost) Impossible Integrals, Sums, and Series, Problem Books in Mathematics*. Springer International Publishing.

[11] Borwein, J. M., & Bradley, D. M. (2006). Thirty-two Goldbach variations. *International Journal of Number Theory, 2*(1), 65-103.

[12] Borwein, D., & Borwein, J. M. (1995). On an intriguing integral and some series related to $\zeta(4)$. *Proceedings of the American Mathematical Society, 123*(4), 1191-1198.

[13] Guo, B. N., Lim, D., & Qi, F. (2021). Maclaurin series expansions for powers of inverse (hyperbolic) sine, for powers of inverse (hyperbolic) tangent, and for incomplete gamma functions, with applications to second kind Bell polynomials and generalized logsine function. *AIMS Mathematics 6*(7), 7494-7517.

[14] Borwein, J. M., & Chamberland, M. (2007). Integer powers of Arcsin. *International Journal of Mathematics and Mathematical Sciences, 2007*, Article ID 19381, 10 pages.

[15] Wolfram alpha. https://www.wolframalpha.com.

[16] Hoffman, M. E. (2019). An odd variant of multiple zeta values. *Communications in Number Theory and Physics, 13*(3), 529-567.

[17] Zagier, D. (2012). Evaluation of the multiple zeta values $\zeta(2,\cdots,2,3,2,\cdots,2)$. *Annals of Mathematics*, *175*, 977-1000.