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Solving the Space -Time Fractional RLW and MRLW Equations Using Modified Extended Tanh Method with the Riccati Equation

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Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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Abstract

In the present paper, a traveling wave solution has been established using the modified extended tanh method for space-time fractional nonlinear partial differential equations. We used this method to find exact solutions for different types of the space-time fractional nonlinear partial differential equations such as space-time fractional regularized long wave equation (RLWE) and space-time fractional modified regularized long wave equation (MRLW) which are the important soliton equations. Both equations are reduced to ordinary differential equations by using of fractional complex transform and properties of modified Riemann-Liouville derivative.

Keywords: Fractional RLW and fractional MRLW equations; the modified extended tanh method.

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1 Introduction

To generalize the classical differential equations with integer orders, fractional differential equations have been represented where the fractional differential equations have been played an important role in different research areas. Specially in mechanics, signal processing, engineering, stochastic, biology, plasma physics, electricity, electrochemistry, dynamical system, control theory, systems identification, economics, and finance.

The numerical solutions of nonlinear systems and nonlinear equations are important in applied science. In the literature different equations have been solved using different methods for example, Hirota-Satsuma coupled KDV equation by Raslan et al. [1], Hirota equation by Raslan et al. [2], generalized long wave equation system by El- Danaf et al. [3,4], coupled-BBM system has been solved by Raslan et al. [5-7], and Coupled Burgers' equations has been studied by Ali et al. [8] and by Raslan et al. [9].

Finding exact and approximate solutions to fractional differential equations is an important task. Powerful and reliable methods have been proposed to obtain the exact solutions of fractional differential equations, such as first integral method [10-15], ansatz method [16-19], exp-function method [20-24], functional variable method [25-28], Kudryashov method [29,30], exp $(-\phi(\xi))-$ expansion method[31,32] , and extended (G'/G)-expansion method [33,34].

Soliton theory is an important areas of research in ocean dynamics, optics, plasma physics, fluid dynamics, semiconductors and engineering. In these areas, studying solitary waves attracts researcher's attention. In the recent years, several studies have been introduced in the field of space time fractional differential equations such as, K. Hosseini et al. [35,36], M. Eslami [37], and M. Kaplan et al. [38].

This paper is organized as follows: In Section 2, the modified Riemann– Liouville derivative is described. In section 3, we illustrate how fractional differential equations are converted into integer-order differential equations. In Section 4, we apply the proposed modified extended tanh method to get the exact solutions for the space– time fractional RLW and MRLW equations. Conclusions are presented in Section 5.

2 Jumarie's Modified Riemann-Liouville Derivative and its Properties

The Jumarie's modified Riemann-Liouville derivative of order α the continuous function $f : R \to R$ is defined as follows [39].

$$D_x^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dx} \displaystyle\int_0^x (x-t)^{-\alpha} \left(f(t) - f(0) \right) dt, & 0 < \alpha < 1, \\ \left(f^{n}(x) \right)^{(\alpha-n)}, \; n \le \alpha < n+1, \; n \ge 1. \end{cases} \tag{1}$$

where $\Gamma(x)$ is the Gamma function which is defined as

$$\Gamma(x) = \int_0^x e^{-t} t^{x-1} dt.$$

Some useful properties of the Jumarie's modified Riemann-Liouville derivative are listed below.

Property 1.

$$D_x^{\alpha} x^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} x^{r-\alpha}. \tag{2}$$

Property 2.

$$D_x^{\alpha}\left(a\, f(x) + b\, g(x)\right) = a D_x^{\alpha} f(x) + b D_x^{\alpha} g(x), \tag{3}$$

where a and b are constants.

Property 3.

$$D_x^{\alpha} f(\xi) = \frac{df}{d\xi} D_x^{\alpha}(\xi), \tag{4}$$

where $\xi = g(x)$.

For other properties, see [40].

3 The Properties of the Methodology

To show the basic idea of our method, consider the following nonlinear fractional differential equation

$$F\left(u, D_t^{\alpha_1}u, D_x^{\alpha_2}u, D_{tt}^{2\alpha_1}u, D_{xx}^{2\alpha_2}u, D_t^{\alpha_1}D_x^{\alpha_2}u, \ldots\right), \qquad 0 < \alpha_1, \alpha_2 < 1. \tag{5}$$

Applying the fractional complex transformation

$$u(x,t) = f(\xi) \quad , \quad \xi = \frac{k}{\Gamma(1+\alpha_2)} x^{\alpha_2} - \frac{c}{\Gamma(1+\alpha_1)} t^{\alpha_1} - x_0,$$

where k and c are nonzero constants and x_0 is arbitrary constant, converts (5) into an integer order nonlinear ordinary differential equations as follows:

$$H(f', f'', f''', \ldots) = 0, \tag{6}$$

where the derivatives are with respect to ξ. It is assumed that the solutions of (6) is presented as a finite series, say

$$f(\xi) = a_0 + \sum_{n=1}^{N}\left(a_n \phi^n(\xi) + b_n \phi^{-n}(\xi)\right), \tag{7}$$

where a_n, b_n, $n = 1, 2, \ldots\ldots, N$ are constants that can be computed and $\phi^n(\xi)$ satisfies the Riccati equation

$$\phi' = b + \phi^2 \tag{8}$$

where b is a constant, Eq. (8) admits several types of solutions:

(i) If $b < 0$, then

$$\phi = -\sqrt{-b}\tanh\left(\sqrt{-b}\xi\right), \text{ or } \phi = -\sqrt{-b}\coth\left(\sqrt{-b}\xi\right)$$

(ii) If $b > 0$, then

$$\phi = \sqrt{b}\tan\left(\sqrt{b}\xi\right), \text{ or } \phi = -\sqrt{b}\cot\left(\sqrt{b}\xi\right)$$

(iii) If $b = 0$, then

$$\phi = \frac{-1}{\xi}.$$

The value of N is usually determined by balancing the linear and nonlinear terms of highest orders in (5). Substituting Eq. (7) and its necessary derivatives, for example

$$f' = \sum_{n=1}^{N}\left(a_n n \phi^{n-1}\left(b+\phi^2\right) - b_n n \phi^{-n-1}\left(b+\phi^2\right)\right),$$

$$f'' = \sum_{n=1}^{N}\begin{pmatrix} a_n n(n-1)\phi^{n-2}\left(b+\phi^2\right)^2 + 2na_n\ \phi^n\left(b+\phi^2\right) + b_n n(n+1)\phi^{-n-2}\left(b+\phi^2\right)^2 - \\ 2b_n n \phi^{-n}\left(b+\phi^2\right) \end{pmatrix},$$

into (5) gives

$$P(\phi(\xi)) = 0, \tag{9}$$

where $P(\phi(\xi))$ is a polynomial in $\phi(\xi)$. By equating the coefficient of each power of $\phi(\xi)$ in (9) to zero, a system of algebraic equations will be obtained whose solution yields the exact solutions of (5).

4 Application

Using the modified extended tanh method, the exact solutions of the space-time fractional RLW and MRLW equations are constructed.

4.1 The space-time fractional RLW equation

Consider the following problem: Find a function $u(x,t)$ satisfying the space-time fractional RLWE in the form,

$$D_t^{\alpha}u(x,t) + D_x^{\alpha}u(x,t) + \varepsilon D_x^{\alpha}u^2(x,t) - \mu D_{xxt}^{3\alpha}u(x,t) = 0, \tag{10}$$

where ε, μ are real parameters applying the fractional complex transformation

$$u(x,t)= f(\xi), \qquad \xi = \frac{k}{\Gamma(1+\alpha)}x^{\alpha} - \frac{c}{\Gamma(1+\alpha)}t^{\alpha} - x_0. \tag{11}$$

From (11) we get,

$$D_t^{\alpha}u = -c\,f'(\xi), \quad D_x^{\alpha}u = k\,f'(\xi), \quad D_{xx}^{2\alpha}u = k^2 f''(\xi), \quad D_{xxt}^{3\alpha}u = -ck^2 f'''(\xi), \ldots,$$

We converts (10) into an integer order nonlinear ordinary differential equation as the following

$$(k-c)f' + \varepsilon k\left(f^2\right)' + \mu ck^2 f''' = 0, \tag{12}$$

integrating (12) once with respect to ξ, yields

$$(k-c)\,f + \varepsilon k\ f^2 + \mu ck^2 f'' = 0, \tag{13}$$

where the integrating constant is considered to be zero.

4.1.1 Exact solutions of the space-time fractional RLW equation using the modified extended tanh method

Balancing f'' and f^2 in (13) results $N+2=2N$, and so $N=2$. This offers a truncated series as the following form

$$f(\xi) = a_0 + a_1\phi(\xi) + a_2\phi^2(\xi) + b_1\phi^{-1}(\xi) + b_2\phi^{-2}(\xi), \tag{14}$$

by substituting (14) into (13) and equating the coefficient of each power of $\phi(\xi)$ to zero. We derive a system of algebraic equations as follows

$$\begin{aligned}
&\varepsilon k\ a_0^{\,2} + (k-c)a_0 + 2\varepsilon k\ a_1 b_1 + 2\varepsilon k\ a_2 b_2 + 2\mu c k^2 b^2 a_2 + 2\mu c k^2 b_2 = 0,\\
&(k-c)a_1 + 2\varepsilon k\ a_0 a_1 + 2\varepsilon k\ a_2 b_1 + 2\mu c k^2 b a_1 = 0,\\
&\varepsilon k\ a_1^{\,2} + (k-c)a_2 + 2\varepsilon k\ a_0 a_2 + 8\mu c k^2 b a_2 = 0,\\
&(k-c)b_1 + 2\varepsilon k\ a_0 b_1 + 2\varepsilon k\ a_1 b_2 + 2\mu c k^2 b b_1 = 0,\\
&\varepsilon k\ b_1^{\,2} + (k-c)b_2 + 2\varepsilon k\ a_0 b_2 + 8\mu c k^2 b b_2 = 0,\\
&2\varepsilon k\ a_2 a_1 + 2\mu c k^2 a_1 = 0,\\
&\varepsilon k\ a_2^{\,2} + 6\mu c k^2 a_2 = 0,\\
&2\varepsilon k\ b_1 b_2 + 2\mu c k^2 b^2 b_1 = 0,\\
&\varepsilon k\ b_2^{\,2} + 6\mu c k^2 b^2 b_2 = 0.
\end{aligned}$$

solving the above system yields.

Case 1.

$$a_1 = b_1 = b_2 = 0, \quad a_0 = \frac{(c-k) - 2\sqrt{(c-k)^2}}{2k\varepsilon}, \quad a_2 = -\frac{6ck\mu}{\varepsilon}, \quad b = \frac{\sqrt{1 + \frac{c^2}{k^2} - \frac{2c}{k}}}{4ck\mu}.$$

hence, the solution is formed as:

$$u_1(x,t) = \frac{(c-k) - 2\sqrt{(c-k)^2}}{2k\varepsilon} - \frac{6ck\mu}{\varepsilon} b \tan^2\left(\sqrt{b}\xi\right),$$

$$u_2(x,t) = \frac{(c-k) - 2\sqrt{(c-k)^2}}{2k\varepsilon} - \frac{6ck\mu}{\varepsilon} b \cot^2\left(\sqrt{b}\xi\right),$$

$$\xi = \frac{k}{\Gamma(1+\alpha)} x^\alpha - \frac{c}{\Gamma(1+\alpha)} t^\alpha - x_0, \; b = \frac{\sqrt{1 + \frac{c^2}{k^2} - \frac{2c}{k}}}{4ck\mu}.$$

Case 2.

$$a_1 = b_1 = b_2 = 0, \quad a_0 = \frac{(c-k) + 2\sqrt{(c-k)^2}}{2k\varepsilon}, \quad a_2 = -\frac{6ck\mu}{\varepsilon}, \quad b = -\frac{\sqrt{1 + \frac{c^2}{k^2} - \frac{2c}{k}}}{4ck\mu}.$$

thus, the solution is formed as:

$$u_3(x,t) = \frac{(c-k) + 2\sqrt{(c-k)^2}}{2k\varepsilon} + \frac{6ck\mu}{\varepsilon} b \tanh^2\left(\sqrt{-b}\xi\right),$$

$$u_4(x,t) = \frac{(c-k) + 2\sqrt{(c-k)^2}}{2k\varepsilon} + \frac{6ck\mu}{\varepsilon} b \coth^2\left(\sqrt{-b}\xi\right),$$

$$\xi = \frac{k}{\Gamma(1+\alpha)} x^\alpha - \frac{c}{\Gamma(1+\alpha)} t^\alpha - x_0, \; b = -\frac{\sqrt{1 + \frac{c^2}{k^2} - \frac{2c}{k}}}{4ck\mu}.$$

Case 3.

$$a_1 = b_1 = 0, \quad a_0 = \frac{2(c-k) - \sqrt{(c-k)^2}}{4k\varepsilon}, \quad b_2 = -\frac{3\left(c^2 - 2ck + k^2\right)}{128k^3 c\mu\varepsilon},$$

$$a_2 = -\frac{6ck\mu}{\varepsilon}, \; b = \frac{\sqrt{1 + \frac{c^2}{k^2} - \frac{2c}{k}}}{16ck\mu}.$$

therefore, the solution is formed as:

$$u_5(x,t)=\frac{2(c-k)-\sqrt{(c-k)^2}}{4k\varepsilon}-\frac{6ck\mu}{\varepsilon}b\tan^2\left(\sqrt{b}\xi\right)-\frac{3\left(c^2-2ck+k^2\right)}{128k^3c\mu\varepsilon b}\cot^2\left(\sqrt{b}\xi\right),$$

$$u_6(x,t)=\frac{2(c-k)-\sqrt{(c-k)^2}}{4k\varepsilon}-\frac{6ck\mu}{\varepsilon}b\cot^2\left(\sqrt{b}\xi\right)-\frac{3\left(c^2-2ck+k^2\right)}{128k^3c\mu\varepsilon b}\tan^2\left(\sqrt{b}\xi\right),$$

$$\xi=\frac{k}{\Gamma(1+\alpha)}x^\alpha-\frac{c}{\Gamma(1+\alpha)}t^\alpha-x_0,\ b=\frac{\sqrt{1+\frac{c^2}{k^2}-\frac{2c}{k}}}{16ck\mu}.$$

Case 4.

$$a_1=b_1=0,\quad a_0=\frac{2(c-k)+\sqrt{(c-k)^2}}{4k\varepsilon},\quad b_2=-\frac{3\left(c^2-2ck+k^2\right)}{128k^3c\mu\varepsilon},$$

$$a_2=-\frac{6ck\mu}{\varepsilon},\ b=-\frac{\sqrt{1+\frac{c^2}{k^2}-\frac{2c}{k}}}{16ck\mu}.$$

hence, the solution is formed as:

$$u_7(x,t)=\frac{2(c-k)+\sqrt{(c-k)^2}}{4k\varepsilon}+\frac{6ck\mu}{\varepsilon}b\tanh^2\left(\sqrt{-b}\xi\right)+\frac{3\left(c^2-2ck+k^2\right)}{128k^3c\mu\varepsilon b}\coth^2\left(\sqrt{-b}\xi\right),$$

$$u_8(x,t)=\frac{2(c-k)+\sqrt{(c-k)^2}}{4k\varepsilon}+\frac{6ck\mu}{\varepsilon}b\coth^2\left(\sqrt{-b}\xi\right)+\frac{3\left(c^2-2ck+k^2\right)}{128k^3c\mu\varepsilon b}\tanh^2\left(\sqrt{-b}\xi\right),$$

$$\xi=\frac{k}{\Gamma(1+\alpha)}x^\alpha-\frac{c}{\Gamma(1+\alpha)}t^\alpha-x_0,\ b=-\frac{\sqrt{1+\frac{c^2}{k^2}-\frac{2c}{k}}}{16ck\mu}.$$

Case 5.

$$a_1=b_1=a_2=0,\quad a_0=\frac{(c-k)+2\sqrt{(k-c)^2}}{2k\varepsilon},\quad b_2=-\frac{3\left(c^2-2ck+k^2\right)}{8k^3c\mu\varepsilon},$$

$$b=-\frac{\sqrt{\frac{1}{c^2}+\frac{1}{k^2}-\frac{2c}{k}}}{4k\mu}.$$

hence, the solution is formed as:

$$u_7(x,t)=\frac{(c-k)+2\sqrt{(k-c)^2}}{2k\varepsilon}+\frac{3\left(c^2-2ck+k^2\right)}{8k^3c\mu\varepsilon b}\coth^2\left(\sqrt{-b}\xi\right),$$

$$u_8(x,t)=\frac{(c-k)+2\sqrt{(k-c)^2}}{2k\varepsilon}+\frac{3\left(c^2-2ck+k^2\right)}{8k^3c\mu\varepsilon b}\tanh^2\left(\sqrt{-b}\xi\right),$$

$$\xi=\frac{k}{\Gamma(1+\alpha)}x^{\alpha}-\frac{c}{\Gamma(1+\alpha)}t^{\alpha}-x_0,\ b=-\frac{\sqrt{\frac{1}{c^2}+\frac{1}{k^2}-\frac{2c}{k}}}{4k\mu}.$$

Case 6.

$$a_1=b_1=a_2=0,\quad a_0=\frac{(c-k)-2\sqrt{(k-c)^2}}{2k\varepsilon},\quad b_2=-\frac{3\left(c^2-2ck+k^2\right)}{8k^3c\mu\varepsilon},$$

$$b=\frac{\sqrt{\frac{1}{c^2}+\frac{1}{k^2}-\frac{2c}{k}}}{4k\mu}.$$

thus, the solution is formed as:

$$u_7(x,t)=\frac{(c-k)-2\sqrt{(k-c)^2}}{2k\varepsilon}-\frac{3\left(c^2-2ck+k^2\right)}{8k^3c\mu\varepsilon b}\cot^2\left(\sqrt{b}\xi\right),$$

$$u_8(x,t)=\frac{(c-k)-2\sqrt{(k-c)^2}}{2k\varepsilon}-\frac{3\left(c^2-2ck+k^2\right)}{8k^3c\mu\varepsilon b}\tan^2\left(\sqrt{b}\xi\right),$$

$$\xi=\frac{k}{\Gamma(1+\alpha)}x^{\alpha}-\frac{c}{\Gamma(1+\alpha)}t^{\alpha}-x_0,\ b=\frac{\sqrt{\frac{1}{c^2}+\frac{1}{k^2}-\frac{2c}{k}}}{4k\mu}.$$

Note. The solutions that we obtained in this paper, are new and are not shown in other method or in the previous literature.

Now, we plot these solutions at different time levels and different values of $\alpha=0.1,1$, respectively and we can show the motion of solitary waves at Fig. 1.

4.2 The space-time fractional MRLW equation

Now, for the space-time fractional MRLWE consider the following problem: Find the functions $u(x,t)$ satisfying the space-time fractional MRLWE in the form,

$$D_t^{\alpha} u(x,t) + D_x^{\alpha} u(x,t) + \varepsilon D_x^{\alpha} u^3(x,t) - \mu D_{xxt}^{3\alpha} u(x,t) = 0, \tag{15}$$

applying the fractional complex transformation (11) converts (15) into an integer order nonlinear ordinary differential equation as the following

$$(k-c) f' + \varepsilon k \left(f^3\right)' + \mu c k^2 f''' = 0. \tag{16}$$

integrating (16) once with respect to ξ, yields

$$(k-c) f + \varepsilon k\, f^3 + \mu c k^2 f'' = 0. \tag{17}$$

Where the integrating constant is considered to be zero.

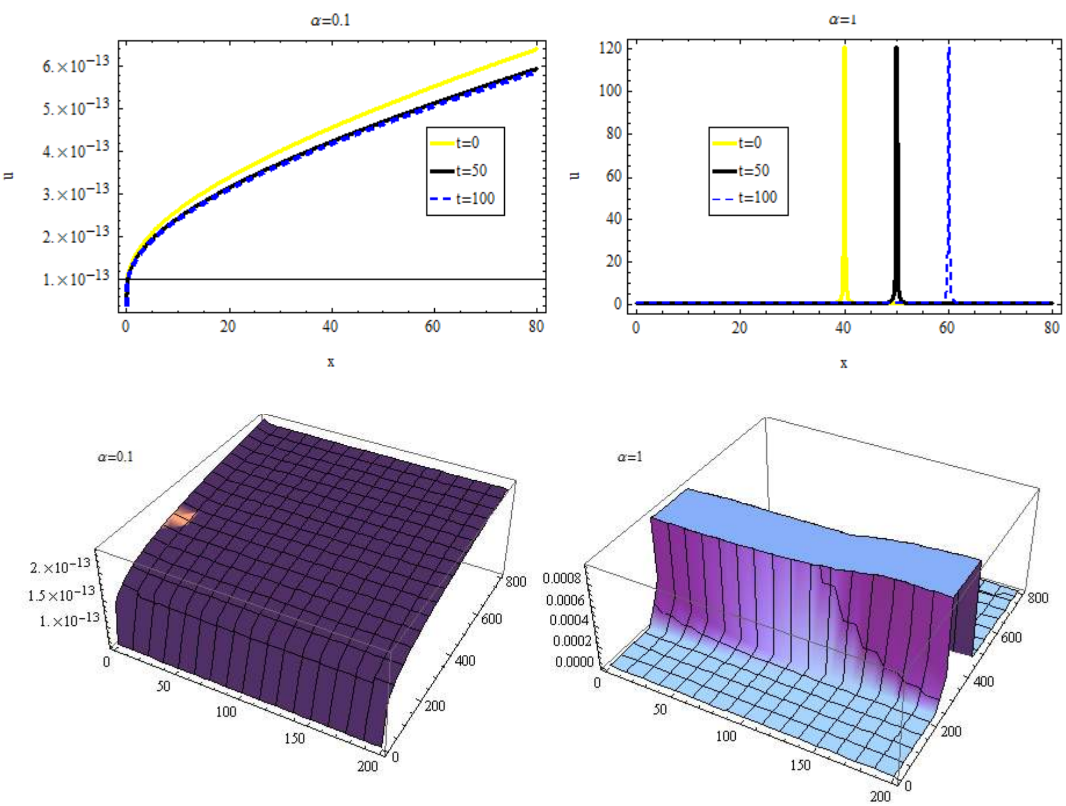

Fig. 1. Exact solutions for the space-time fractional RLWE with $c = 0.1, \varepsilon = 1, \quad 0 \le t \le 100,$ $x_0 = 40, k = 1, \mu = 1, 0 \le x \le 80$**, at different time levels and** $\alpha = 0.1, 1$**, respectively**

4.2.1 Exact solutions of the Space-time fractional MRLW equations using the modified extended tanh method

Balancing f'' and f^3 in (17) results $N + 2 = 3N$, and so $N = 1$. This offers a truncated series as the following form

$$f(\xi) = a_0 + a_1\,\phi(\xi) + b_1\,\phi^{-1}(\xi) \tag{18}$$

By substituting (18) into (17) and equating the coefficient of each power of $\phi(\xi)$ to zero. We derive a system of algebraic equations as follows:

$$\begin{aligned}
&\varepsilon k\ a_0^{\,3} + (k-c)a_0 + 6\varepsilon k\ a_0 a_1 b_1 = 0,\\
&(k-c)a_1 + 3\varepsilon k\ a_0^{\,2} a_1 + 3\varepsilon k\ a_1^{\,2} b_1 + 2\mu c k^2 b a_1 = 0,\\
&3\varepsilon k\ a_1^{\,2} a_0 = 0,\\
&\varepsilon k\ a_1^{\,3} + 2\mu c k^2 a_1 = 0,\\
&(k-c)b_1 + 3\varepsilon k\ a_0^{\,2} b_1 + 3\varepsilon k\ b_1^{\,2} a_1 + 2\mu c k^2 b b_1 = 0,\\
&3\varepsilon k\ b_1^{\,2} a_0 = 0,\\
&\varepsilon k\ b_1^{\,3} + 2\mu c k^2 b^2 b_1 = 0,
\end{aligned}$$

solving the above system, yields.

Case 1.

$$a_0 = 0,\quad a_1 = \mp\frac{i\sqrt{2}\sqrt{c}\sqrt{k}\sqrt{\mu}}{\sqrt{\varepsilon}},\quad b_1 = \pm\frac{i(c-k)}{4\sqrt{2}\sqrt{c}\,k^{3/2}\sqrt{\varepsilon}\sqrt{\mu}},\quad b = \frac{(c-k)}{8ck^2\mu}.$$

thus, the solution is formed as:

$$\begin{aligned}
u_{1,2}(x,t) &= \mp\frac{i\sqrt{2}\sqrt{c}\sqrt{k}\sqrt{\mu}}{\sqrt{\varepsilon}}\sqrt{b}\tan\left(\sqrt{b}\xi\right) \pm \frac{i(c-k)}{4\sqrt{2}\sqrt{c}\,k^{3/2}\sqrt{\varepsilon}\sqrt{\mu}\sqrt{b}}\cot\left(\sqrt{b}\xi\right),\\
u_{3,4}(x,t) &= \pm\frac{i\sqrt{2}\sqrt{c}\sqrt{k}\sqrt{\mu}}{\sqrt{\varepsilon}}\sqrt{b}\cot\left(\sqrt{b}\xi\right) \mp \frac{i(c-k)}{4\sqrt{2}\sqrt{c}\,k^{3/2}\sqrt{\varepsilon}\sqrt{\mu}\sqrt{b}}\tan\left(\sqrt{b}\xi\right),\\
\xi &= \frac{k}{\Gamma(1+\alpha)}x^{\alpha} - \frac{c}{\Gamma(1+\alpha)}t^{\alpha} - x_0,\ b = \frac{(c-k)}{8ck^2\mu}.
\end{aligned}$$

Case 2.

$$a_0 = 0,\quad a_1 = \mp\frac{i\sqrt{2}\sqrt{c}\sqrt{k}\sqrt{\mu}}{\sqrt{\varepsilon}},\quad b_1 = \pm\frac{i(c-k)}{2\sqrt{2}\sqrt{c}\,k^{3/2}\sqrt{\varepsilon}\sqrt{\mu}},\quad b = -\frac{(c-k)}{4ck^2\mu}.$$

hence, the solution is formed as:

$$u_{5,6}(x,t)=\pm\frac{i\sqrt{2}\sqrt{c}\sqrt{k}\sqrt{\mu}}{\sqrt{\varepsilon}}\sqrt{-b}\tanh\left(\sqrt{-b}\xi\right)\mp\frac{i(c-k)}{2\sqrt{2}\sqrt{c}\,k^{3/2}\sqrt{\varepsilon}\sqrt{\mu}\sqrt{-b}}\coth\left(\sqrt{-b}\xi\right),$$

$$u_{7,8}(x,t)=\pm\frac{i\sqrt{2}\sqrt{c}\sqrt{k}\sqrt{\mu}}{\sqrt{\varepsilon}}\sqrt{-b}\coth\left(\sqrt{-b}\xi\right)\mp\frac{i(c-k)}{2\sqrt{2}\sqrt{c}\,k^{3/2}\sqrt{\varepsilon}\sqrt{\mu}\sqrt{-b}}\tanh\left(\sqrt{-b}\xi\right),$$

$$\xi=\frac{k}{\Gamma(1+\alpha)}x^{\alpha}-\frac{c}{\Gamma(1+\alpha)}t^{\alpha}-x_0,\, b=-\frac{(c-k)}{4c\,k^2\mu}.$$

Case 3.

$$a_0=a_1=0,\quad b_1=\mp\frac{i(c-k)}{\sqrt{2}\sqrt{c}\,k^{3/2}\sqrt{\varepsilon}\sqrt{\mu}},\ \ b=\frac{(c-k)}{2c\,k^2\mu}.$$

thus, the solution is formed as:

$$u_{9,10}(x,t)=\mp\frac{i(c-k)}{\sqrt{2}\sqrt{c}\,k^{3/2}\sqrt{\varepsilon}\sqrt{\mu}\sqrt{b}}\cot\left(\sqrt{b}\xi\right),$$

$$u_{11,12}(x,t)=\pm\frac{i(c-k)}{\sqrt{2}\sqrt{c}\,k^{3/2}\sqrt{\varepsilon}\sqrt{\mu}\sqrt{b}}\tan\left(\sqrt{b}\xi\right),$$

$$\xi=\frac{k}{\Gamma(1+\alpha)}x^{\alpha}-\frac{c}{\Gamma(1+\alpha)}t^{\alpha}-x_0,\, b=\frac{(c-k)}{2c\,k^2\mu}.$$

Case 4.

$$a_0=b_1=0,\quad a_1=\mp\frac{i\sqrt{2}\sqrt{c}\sqrt{k}\sqrt{\mu}}{\sqrt{\varepsilon}},\ \ b=\frac{c-k}{2k^2\mu}.$$

hence, the solution is formed as:

$$u_{13,14}(x,t)=\mp\frac{i\sqrt{2}\sqrt{c}\sqrt{k}\sqrt{\mu}}{\sqrt{\varepsilon}}\sqrt{b}\tan\left(\sqrt{b}\xi\right),$$

$$u_{15,16}(x,t)=\pm\frac{i\sqrt{2}\sqrt{c}\sqrt{k}\sqrt{\mu}}{\sqrt{\varepsilon}}\sqrt{b}\cot\left(\sqrt{b}\xi\right),$$

$$\xi=\frac{k}{\Gamma(1+\alpha)}x^{\alpha}-\frac{c}{\Gamma(1+\alpha)}t^{\alpha}-x_0,\, b=\frac{c-k}{2k^2\mu}.$$

Comparing our results with Melike et al results [41], then it can be seen that our results are quite different.

Now, we can plot these solutions at different time levels and different values of $\alpha = 0.1, 1$, respectively and we can show the motion of solitary waves at Fig. 2.

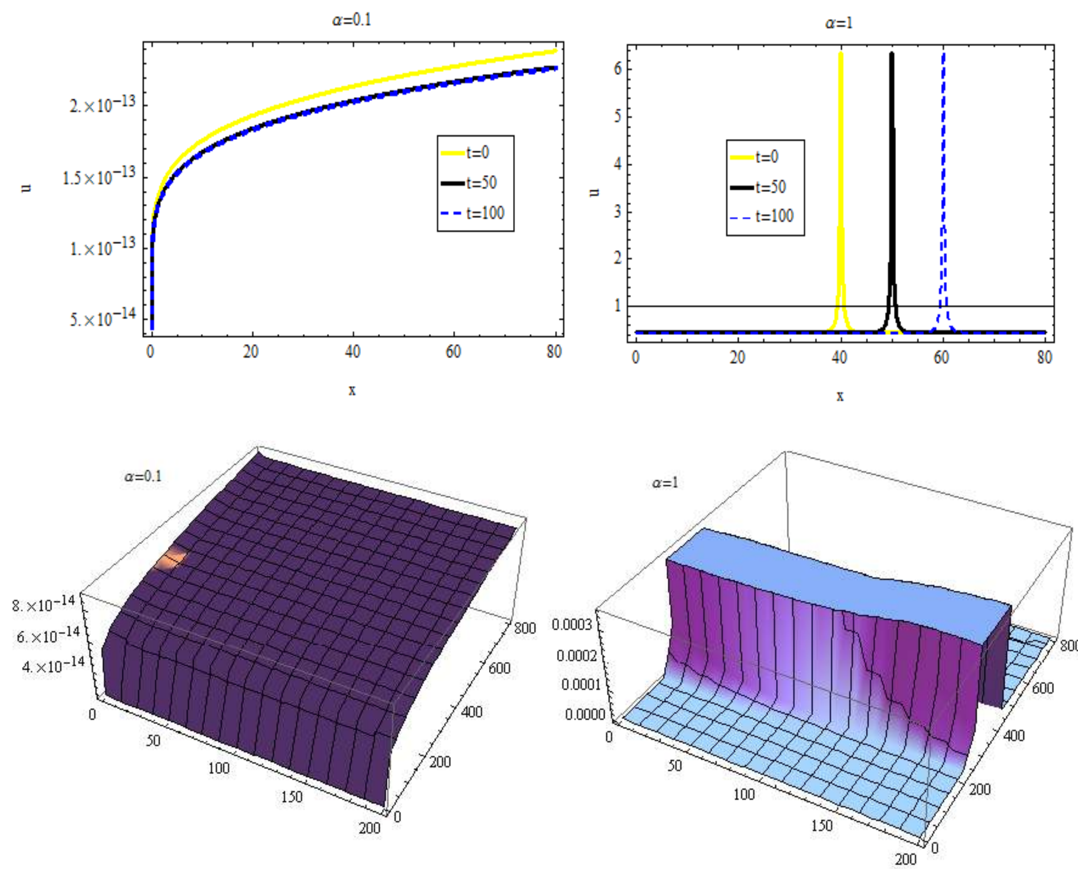

Fig. 2. Exact solutions for the space-time fractional MRLWE with $c = 0.1, \varepsilon = 6,\ 0 \le t \le 100,$ $x_0 = 40, k = 1, \mu = 1, 0 \le x \le 80$ **, at different time levels and** $\alpha = 0.1,\ 1$ **, respectively**

5 Conclusion

The space-time fractional RLW and MRLW equations with modified Riemann-Liouville derivatives proposed by Jumarie were successfully solved in the present paper. To reach this goal, by introducing a fractional complex transformation, original equations were converted into the integer order ordinary differential equations. Then, the effect of the modified extended tanh method was utilized to construct the exact solutions of the resulted equations. The method used, is an efficient and promising technique to handle a wide range of nonlinear fractional differential equations.

Competing Interests

Authors have declared that no competing interests exist.

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