



Regional Boundary Exact Controllability of the Wave Equation by Strategic Actuators on a Polygonal Domain with Cracks

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Authors' contributions

This work was carried out in collaboration among all the authors. Author MTN designed the study and wrote the first draft of the manuscript. The mathematical analysis of the problem was performed by authors CS and OS. Furthermore author CS managed the structural consistency and the English language. While authors CS and OS controlled the calculations and the formulas analysis. All authors managed the literature review, read and approved the final manuscript.

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Abstract

In this work we prove the exact controllability of the wave equation by acting on a strategic zone of the border of a non-convex polygonal domain with crack. Indeed, by combining two methods: that of Grisvard on the exact controllability on domains with corners and that of EL. Jai on the boundary strategic actuators, this exact controllability result has been proven.

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1 Introduction and Statement of The Problem

The exact controllability of distributed systems has seen a marked revival in recent years; thanks to the development of the HUM method (Hilbert Uniqueness Methods) by J.L Lions [1]. It is essentially based on the uniqueness of the properties adapted to the homogeneous equation by a particular choice of controls, the construction of a Hilbert space and a continuous linear application of this space in its dual which is, in fact, an isomorphism which makes it possible to establish exact controllability. This method has yielded significant results for hyperbolic problems Lions [1], Kondratiev [2], Niane [3],[4], Seck [5], [6],...

The presence of a crack in pressure equipment, for obvious safety reasons, to know precisely its degree of harmfulness. When this crack propagates, under cyclic loading, it is important to evaluate and quickly control the evolution of this degree of harmfulness and more concretely the residual life of the cracked structure.

For example, thin plates and shells are widely used in aeronautics. In due to the significant stresses to which the structure of an aircraft in flight is subjected by example, the appearance of small cracks is inevitable.

So, when the controls are in areas with small supports, this method is no longer effective. Moreover, for technical reasons, the multiplier method does not always give the expected results see Grisvard [7], Niane [3].

In this work, we have merged two methods:

- i. From the dynamic system data defined on a domain Ω , we are interested in its controllability only on a region or privileged area of the domain. Starting from classical principles, we develop their adaptations to regional cases in cracked and/or wedge domains.
- ii. For a cracked non-convex polygonal domain, the multiplier method is inoperative because the expression $m(x) \cdot \frac{\partial u}{\partial \nu}$ is not generally square integrable into the edge of each crack tip.

To circumvent these difficulties, controls are posed on strategic zones with small supports over the tips of the cracks and by adding geometrical conditions (lines of concurrent cracks ie $m(x) \cdot \nu > 0$) and the introduction of truncation functions to gain regularity.

1.1 Notations and assumptions

Let Ω a non-empty bounded polygonal domain of \mathbb{R}^2 whose border Γ is the meeting of edges Γ_j for $0 \leq j \leq N$.

We denote S_j (resp S_0) the vertex between Γ_{j-1} and Γ_j for $1 \leq j \leq N$ (resp Γ_N and Γ_0), ω_j : the measure of the interior angle between Γ_{j-1} and Γ_j and ω_0 : the measurement of the interior angle between Γ_N and Γ_0 .

We denote also : $S = \{j \in \{0, \dots, N\} / \omega_j = 2\pi\}$ and assume that $S \neq \emptyset$.

Denote ν_j the unitary normal vector outside Γ_j and τ_j : the unit vector tangent to Γ_j and directed to S_j tips.

Let $0 < \epsilon < \delta < \gamma$ and define the following sets:

$$\left\{ \begin{array}{l} Q_T = \Omega \times]0, T[\\ \Sigma_T = \Gamma \times]0, T[\\ \mathcal{O}_j^T = B(S_j, \gamma) \times]0, T[\\ \Gamma_\delta = \Gamma \setminus \bigcup_{j \in S} [\Gamma \cap B(S_j, \delta)] \\ \Gamma_{\delta\epsilon}^j = \Gamma \cap [B(S_j, \delta) \setminus B(S_j, \epsilon)] \\ \Gamma_\delta(x_0) = \{x \in \Gamma_\delta, (x - x_0) \cdot \nu > 0\} \\ \Sigma_\delta = \Gamma_\delta \times [0, T] \\ \Sigma_\delta(x_0) = \Gamma_\delta(x_0) \times]0, T[\\ \Gamma^* = \Gamma_\delta(x_0) \cup \{S_j, j \in S\} \\ O = \bigcup_{x \in \Gamma^*} B(x, \gamma) \end{array} \right.$$

Also, let η_j : a truncation function such that: $0 \leq \eta_j \leq 1$, $\text{supp}(\eta_j) \subset B(S_j, \frac{3\gamma+\delta}{4})$, $\eta_j = 1$ on $B(S_j, \frac{3\gamma+\delta}{4})$, $m(x) = (x - x_0)$ and $R_0 = R(x_0) = \max \{ |m(x)|, x \in \bar{\Omega} \}$.

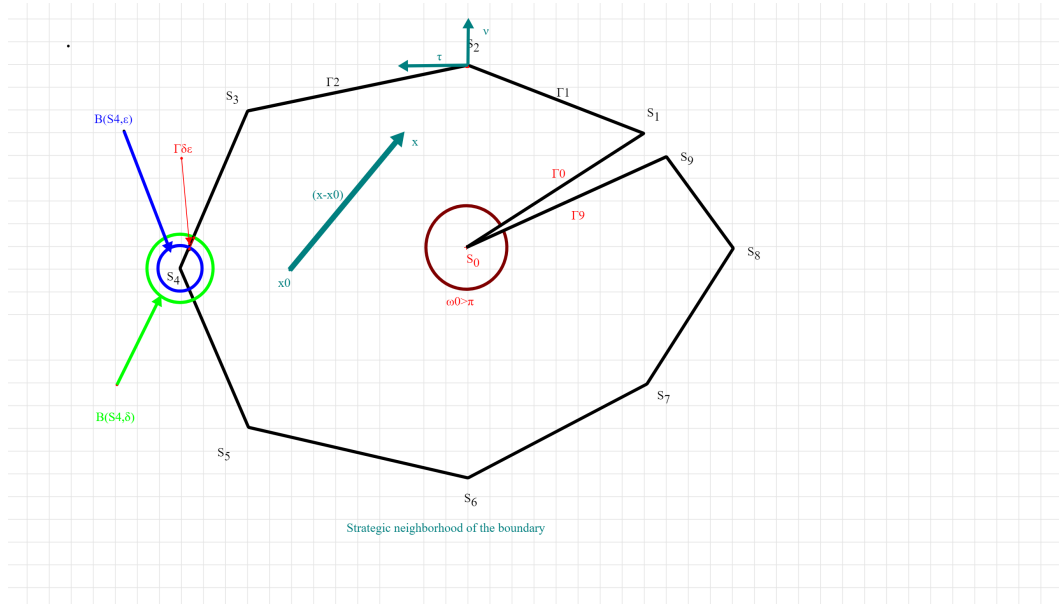


Fig. 1. Domain with crack

1.2 Reminders and statement of the problem

Let $f \in L^2(\Omega)$ and $u \in H_0^1(\Omega)$ the unique solution of Dirichlet problem:

$$\begin{cases} -\Delta u = f \text{ in } Q_T \\ \gamma u = 0 \text{ on } \Gamma_T \end{cases} \quad (1.1)$$

In the space $H = L^2(\Omega)$, we consider A the positive self-adjoint operator defined by the Laplace operator with Dirichlet's condition (1.1):

$$D(A) = \left\{ u \in H_0^1(\Omega); -\Delta u \in L^2(\Omega) \right\}, \\ Au = -\Delta u,$$

for all $u \in D(A)$.

When Ω is regular, we know that $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$.

On the other hand, for the case which interests us $D(A)$ is not contained in $H^2(\Omega)$. However it can be described as $H^2(\Omega) \cap H_0^1(\Omega) \oplus \text{lin}(\{s_1, \dots, s_k\})$ where k denotes the number of interior angles at Ω with an opening greater than π and the s_j are singular functions in the sense that they are in $H_0^1(\Omega)$ but not in $H^2(\Omega)$.

Indeed, if $f \in L^2(\Omega)$, the solution $u \in H_0^1(\Omega)$ of the Dirichlet problem is given by P. Grisvard [4],[9]:

$$u = u_R + \sum_{j \in S} c_j s_j \quad (1.2)$$

$$= u_R + \sum_{j \in S} c_j \sqrt{r_j} \sin\left(\frac{\theta_j}{2}\right) \eta_j \quad (1.3)$$

where $u_R \in H^{\frac{3}{2}+\alpha}(\Omega)$ is the regular part of the solution u , $(c_j)_{j \in S}$ are reals constants (singularity coefficients), s_j are singular functions, $\alpha > 0$ in general $\alpha \in]0, \frac{1}{2}[$ and (r_j, θ_j) corresponding to polar coordinates.

The problem of boundary exact controllability of the wave equation, consists for $T > 0$ fixed and $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, to look for a control $v \in L^2(\Sigma_T)$ such that if u is the solution of the problem defined by:

$$\begin{cases} u'' - \Delta u = 0 \text{ in } Q_T \\ \gamma u = v \text{ on } \Sigma_T \\ u(0) = u_0 \text{ in } \Omega \\ u'(0) = u_1 \text{ in } \Omega \end{cases} \quad (1.4)$$

then $u(T) = u'(T) = 0$.

Indeed, in [1], Lions showed by the HUM method that fact, v can be chosen as the restriction to Σ_T of the normal derivative of a solution u of the homogeneous wave equation with initial conditions $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$.

However, the application of this method in the case of cracked domains or domains with corner generate serious difficulties because $\frac{\partial u}{\partial \nu}$ is generally not integrable square at cracks tip.

Grisvard in [10] and Dauge in [11] overcame this difficulty by substituting for the normal derivative $\frac{\partial u}{\partial \nu}$ the following control: $(x - x_0) \cdot \nu \frac{\partial u}{\partial \nu}$ with $(x - x_0) \cdot \nu = 0$ at each crack tips so that the following relation to make sense:

$$\int_{\Sigma_T} (x - x_0) \cdot \nu \left| \frac{\partial u}{\partial \nu} \right|^2 d\sigma dt \quad (1.5)$$

These "drastic" geometrical conditions impose on the lines of cracks to be concurrent.

And even this condition is often insufficient to establish inequalities of observability (or the equivalence of norms) by the HUM method.

To get around these difficulties (or even remedy them), another condition relating to the position of the cracks with respect to x_0 is imposed, namely $(S_j - x_0) \cdot \tau_j > 0$ for all $j \in S$. These conditions are very restrictive for practical use; So, in order to overcome these difficulties and lift these geometrical conditions, we based ourselves on the work of Niane[12] and El Jai [13],[14], by proposing more boundary control to the neighborhood of crack tips, add border controls with small support on a privileged neighborhood (called strategic) of these vertexes of cracks; and therefore the exact controllability without geometrical conditions has been proven.

1.3 Notions of regional controllability and strategic zone actuators

Definition 1.

The system (1.4) is said to be exactly regionally controllable on a subdomain $\omega \subsetneq \Omega$ (or even exactly ω -controllable) if for all u_d (ideal solution) given in $L^2(\omega)$, there is a control $v \in U$ (admissible set) such that

$$u(., T, v) = u_d.$$

Remark 1.1. i. It should be noted that, in the distributed case, the concept of exact controllability (global or regional) is not suitable and remains very impractical even for an action exercised on the whole domain Ω .

ii. In what will follow of this work, we will not pay attention to the final state of the system on $\Omega \setminus \omega$: these notions have given rise to multiple developments in the literature see Grisvard [9], Dauge [11], Seck [15], [16], Moussaoui [17], Timouyas [18],... .

Definition 2.

Any pair (ω_0, μ_0) is called an actuator where

1. $\omega_0 \subset \Omega$: is the geometric support of the action.
2. $\mu_0 \in L^2(\omega_0)$ is the spatial distribution of the action.

Definition 3.

We say that an actuator is strategic (respectively ω -strategic) if the system which it excites is weakly controllable (respectively weakly ω -controllable).

From these definitions will emerge three major difficulties:

- On the choice of the suitable state space.
- On the number of actuators for the control of the system.
- And on the choice and delimitation of the privileged area known as strategic.

1.4 Integrations formulas by parts on a domain with crack

Starting from the decomposition of the domain and its border (1.1), we have the following fundamental theorem of integrations by parts:

Theorem 1. For all $f \in L^2(\Omega)$ and $u \in H_0^1(\Omega)$ solution of the equation (1) with

$$u = \underbrace{u_R}_{\text{the regular part}} + \underbrace{\sum_{j \in S} c_j \sqrt{r_j} \sin\left(\frac{\theta_j}{2}\right) \eta_j}_{\text{the singular part}}$$

where $u_R \in H^{\frac{3}{2}+\alpha}(\Omega)$ then we have:

$$\begin{aligned} \int_{\Omega} -\Delta u(x-x_0) \cdot \nabla u dx &= -\frac{\pi}{4} \sum_{j \in S} (S_j - x_0) \cdot \tau_j c_j^2 - \frac{1}{2} \int_{\Gamma_{\delta}} (x-x_0) \cdot \nu \left(\frac{\partial u}{\partial \nu}\right)^2 d\sigma \\ &\quad - \frac{1}{2} \sum_{j \in S} \lim_{\epsilon \rightarrow 0} \int_{\Gamma_{\delta\epsilon}^j} (x-x_0) \cdot \nu_j \left(\frac{\partial u}{\partial \nu_j}\right)^2 d\sigma \end{aligned} \quad (1.6)$$

□

Proof. Let $0 < \epsilon < \delta$; Let us carry out the integrations by parts on Ω_{ϵ} not containing the open balls $B(S_j, \epsilon)$.

Let so $\Omega_{\epsilon} = \Omega \setminus \cup_{j \in S} B(S_j, \epsilon)$ and $\gamma_j = \Omega \cap \partial B(S_j, \epsilon)$.

Then we have the border decomposition following

$$\Gamma = \Gamma_{\delta} \cup \cup_{j \in S} \Gamma_{\delta\epsilon}^j \cup \cup_{j \in S} \gamma_j^j$$

By integrating by parts (Green in cracked domains see Grisvard [10],[8]) on these partitions we get

$$\begin{aligned} \int_{\Omega} -\Delta u \cdot m(x) \cdot \nabla u dx &= -\frac{1}{2} \int_{\Gamma_{\delta}} (x-x_0) \cdot \nu \left(\frac{\partial u}{\partial \nu}\right)^2 d\sigma \\ &\quad - \frac{1}{2} \sum_{j \in S} \int_{\Gamma_{\delta\epsilon}^j} (x-x_0) \cdot \nu_j \left(\frac{\partial u}{\partial \nu_j}\right)^2 d\sigma + 2(x-x_0) \cdot \tau_j \frac{\partial u}{\partial \nu_j} \frac{\partial u}{\partial \tau_j} - (x-x_0) \cdot \nu_j \left(\frac{\partial u}{\partial \tau_j}\right)^2 \\ &\quad - \frac{1}{2} \sum_{j \in S} \int_{\gamma_j^j} \left[(x-x_0) \nu_j \left(\frac{\partial u}{\partial \nu_j}\right)^2 d\sigma + 2(x-x_0) \cdot \tau_j \frac{\partial u}{\partial \nu_j} \frac{\partial u}{\partial \tau_j} - (x-x_0) \cdot \nu_j \left(\frac{\partial u}{\partial \tau_j}\right)^2 \right] \end{aligned} \quad (1.7)$$

So

$$\int_{\Omega} -\Delta u(x-x_0) \cdot \nabla u dx = I_1 + I_2 + I_3 \quad (1.8)$$

where

$$I_1 = -\frac{1}{2} \int_{\Gamma_{\delta}} (x-x_0) \cdot \nu \left(\frac{\partial u}{\partial \nu}\right)^2 d\sigma \quad (1.9)$$

$$I_2 = -\frac{1}{2} \sum_{j \in S} \int_{\Gamma_{\delta\epsilon}^j} (x-x_0) \cdot \nu_j \left(\frac{\partial u}{\partial \nu_j}\right)^2 d\sigma + 2(x-x_0) \cdot \tau_j \frac{\partial u}{\partial \nu_j} \frac{\partial u}{\partial \tau_j} - (x-x_0) \cdot \nu_j \left(\frac{\partial u}{\partial \tau_j}\right)^2 \quad (1.10)$$

and

$$I_3 = -\frac{1}{2} \sum_{j \in S} \int_{\gamma_j^j} \left[(x-x_0) \nu_j \left(\frac{\partial u}{\partial \nu_j}\right)^2 d\sigma + 2(x-x_0) \cdot \tau_j \frac{\partial u}{\partial \nu_j} \frac{\partial u}{\partial \tau_j} - (x-x_0) \cdot \nu_j \left(\frac{\partial u}{\partial \tau_j}\right)^2 \right] \quad (1.11)$$

By direct calculations we have

$$I_3 = \frac{\pi}{4} \sum_{j \in S} (S_j - x_0) \cdot \tau_j c_j^2 + o(\epsilon). \quad (1.12)$$

with $\lim_{\epsilon \rightarrow 0} o(\epsilon) = 0$.

Knowing that $-\Delta u.m(x).\nabla u \in L^2(\Omega)$, we have

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega_\epsilon} -\Delta u(x - x_0).\nabla u dx = \int_{\Omega} -\Delta u(x - x_0).\nabla u dx \tag{1.13}$$

We deduce from this equality that the limit of I_1 exists and this completes the proof. \square

2 Estimates and Norm Equivalences on Strategic Open Sets

2.1 Energy norms on non convex domains

In this part, one establishes via the formula of integration by parts of Theorem 1.4, the equivalence of the norms which result from the judicious choice of the open sets in privileged zones known as strategic of the border and by the method HUM from Lions [1].

Consider a neighborhood \mathcal{V} of Γ^* and let $\mathcal{O} = \mathcal{V} \cap \Omega$.

Remark 2.1. We see that the open \mathcal{O} as previously defined 1.1 is a privileged area of the edge in the sense that it is not only a neighborhood of $\Gamma_\delta(x_0)$, but also for the crack tips.

Theorem 2.1. *There exist $T_0 > 0$ such that for all $T > T_0$, the norm \mathcal{N}_1 defined on $L^2(\Omega) \times H^{-1}(\Omega)$ by*

$$\mathcal{N}_1(u_0, u_1) = \left\{ \int_0^T \int_{\mathcal{O}} \|u\|^2 \right\}^{\frac{1}{2}} \tag{2.1}$$

where u is solution of equation (1.1), is equivalent to the norm defined by

$$E_{0,-1}(u_0, u_1) = \frac{1}{2} \left(\|u_0\|_{L^2(\Omega)}^2 + \|u_1\|_{H^{-1}(\Omega)}^2 \right) \tag{2.2}$$

Lemma 2.2. *There exists $T_0 > 0$ such that for all $T > T_0$, the norm \mathcal{N}_2 defined on $H_0^1(\Omega) \times L^2(\Omega)$ by*

$$\mathcal{N}_2(u_0, u_1) = \left\{ \int_0^T \int_{\mathcal{O}} \|u\|^2 + \|u'\|_{H^{-1}(\Omega)}^2 \right\}^{\frac{1}{2}} \tag{2.3}$$

is an equivalent norm to the energy norm defined by

$$E_{0,1}(u_0, u_1) = \frac{1}{2} \left(\|\nabla u_0\|^2 + \|u_1\|^2 \right) \tag{2.4}$$

Proof. Let $(.,.)$ the scalar product defined on $L^2(\Omega)$.

Let's develop $\int_{\Omega} (u'' - \Delta u)(x - x_0).\nabla u dx dt$ by using the integration formulas by parts of Theorem 1.4, we obtain

$$\begin{aligned} \int_{\Omega} (u'' - \Delta u)(x - x_0).\nabla u dx dt &= (u', (x - x_0).\nabla u)|_0^T + \frac{1}{2} \int_{Q_T} \|u'\|^2 dx dt + \frac{\pi}{4} \sum_{j \in S} (S_j - x_0).\tau_j \int_0^T c_j^2 dt \\ &\quad - \frac{1}{2} \int_0^T \int_{\Gamma_\delta(x_0)} (x - x_0).\nu \left(\frac{\partial u}{\partial \nu} \right) d\sigma dt - \frac{1}{2} \sum_{j \in S} \lim_{\epsilon \rightarrow 0} \int_0^T \int_{\Gamma_{\delta\epsilon}^j} (x - x_0).\nu_j \left(\frac{\partial u}{\partial \nu_j} \right) d\sigma dt = 0 \end{aligned}$$

so, there exist $T_0^1 > 0$ such that

$$\begin{aligned}
 & -\frac{\pi}{4} \sum_{j \in S} (S_j - x_0) \cdot \tau_j \int_0^T c_j^2 dt + \frac{1}{2} \int_0^T \int_{\Gamma_\delta(x_0)} (x - x_0) \cdot \nu \left(\frac{\partial u}{\partial \nu} \right) d\sigma dt \\
 & + \frac{1}{2} \sum_{j \in S} \lim_{\epsilon \rightarrow 0} \int_0^T \int_{\Gamma_{\delta\epsilon}^j} (x - x_0) \cdot \nu_j \left(\frac{\partial u}{\partial \nu_j} \right) d\sigma dt \geq (T - T_0^1) E_{0,1}(u_0, u_1)
 \end{aligned} \tag{2.5}$$

Now let's assume specifically:

$$\mathcal{O}_2 = \cup_{x \in \Gamma^*} B(x, \frac{\delta + \gamma}{2}), \quad \mathcal{O}_3 = \cup_{x \in \Gamma^*} B(x, \frac{\delta + 3\gamma}{4})$$

and, consider a cut off function η such that $supp(\eta) \subset \mathcal{O}_3$ and $\eta|_{\mathcal{O}_2} = 1$.

By developing the expression $\int_\Omega (u'' - \Delta u) \eta(x - x_0) \cdot \nabla u dx dt = 0$ and by noting that by construction that: $\eta = 1$ on $\Gamma_{\epsilon\delta}^j$ and also on $\Gamma_\delta(x_0)$, we obtain

$$\begin{aligned}
 & -\frac{\pi}{4} \sum_{j \in S} (S_j - x_0) \cdot \tau_j \int_0^T c_j^2 dt + \frac{1}{2} \int_0^T \int_{\Gamma_\delta(x_0)} (x - x_0) \cdot \nu \left(\frac{\partial u}{\partial \nu} \right) d\sigma dt \\
 & + \frac{1}{2} \sum_{j \in S} \lim_{\epsilon \rightarrow 0} \int_0^T \int_{\Gamma_{\delta\epsilon}^j} (x - x_0) \cdot \nu_j \left(\frac{\partial u}{\partial \nu_j} \right) d\sigma dt \\
 = & \int_{Q_T} ((x - x_0) \cdot \nabla u) (\nabla \eta \cdot \nabla u) dx dt - \frac{1}{2} \int_{Q_T} div(\eta(x - x_0)) |u'|^2 dx dt + \int_{Q_T} \eta |\nabla u|^2 dx dt
 \end{aligned} \tag{2.6}$$

$$+ (u', (x - x_0) \cdot \nabla u)|_0^T + \frac{1}{2} \int_{Q_T} div(\eta(x - x_0)) |u'|^2 dx dt \tag{2.7}$$

Thus, we deduce from the estimate (2.5) and the equality (2.7) that there exists $T_0^2 > 0$ such that

$$\int_0^T \int_{\mathcal{O}_3} (|\nabla u|^2 + |u'|^2) dx dt \geq (T - T_0^2) E_0(u_0, u_1). \tag{2.8}$$

So, there exists a constant $K(\mathcal{O}_2) > 0$ such that

$$\int_{Q_T} \eta |\nabla u|^2 dx dt \leq 2R_0 E_0(u_0, u_1) + K(\mathcal{O}_2) \int_0^T \int_{\mathcal{O}_2} (|u|^2 + |u'|^2) dx dt \tag{2.9}$$

where the relations (20) and (21) complete the proof. \square

2.2 Norm equivalences in non convex domains

Before the main result, let also the fundamental following lemma

Lemma 2.3. *There exists a constant $C > 0$ such that*

$$\int_0^T \int_{\mathcal{O}} |u|^2 dx dt \leq C \int_0^T \int_{\mathcal{O}} |u'|^2 dx dt \tag{2.10}$$

Proof. Reasoned by the absurd.

Assume (2.10) is false ie there is a sequence $(u_{0n}, u_{1n}) \in H_0^1(\Omega) \times L^2(\Omega)$ such that

$$\int_0^T \int_{\mathcal{O}} |u|^2 dx dt = 1 \quad \text{and} \quad n \int_0^T \int_{\mathcal{O}} |u'|^2 dx dt \leq 1. \tag{2.11}$$

According to the Lemma 2.2 and the relation (2.11), the sequence (u_{0n}, u_{1n}) is bounded in $H_0^1(\Omega) \times L^2(\Omega)$. Thus, it converges weakly towards (u_0, u_1) in this space.

By the fact that the injection $H_0^1(\Omega) \times L^2(\Omega) \hookrightarrow L^2(\Omega) \times H^{-1}(\Omega)$ is compact, we deduce that the sequence (u_{0n}, u_{1n}) converges strongly towards (u_0, u_1) in $L^2(\Omega) \times H^{-1}(\Omega)$.

Also, for $u \in \mathcal{C}(0, T; H_0^1(\Omega)) \cap \mathcal{C}^1(0, T; L^2(\Omega))$ solution of the homogeneous wave equation (1.1), we have

$$\int_0^T \int_{\mathcal{O}} \|u_n - u\|^2 dxdt \leq K_1 \left(\|u_{0n} - u_0\|_{L^2(\Omega)}^2 + \|u_{1n} - u_1\|_{H^{-1}(\Omega)}^2 \right) \quad (2.12)$$

This proves that the sequence (u_n) converges strongly towards $u \in L^2(0, T; L^2(\Omega))$. Besides, we had

$$\int_0^T \int_{\mathcal{O}} \|u\|^2 dxdt = 1 \quad (2.13)$$

Furthermore, according to the absurdity hypothesis (2.11), the sequence $(u'_n)_{n \in \mathbb{N}}$ converges to 0 in $L^2(0, T; L^2(\mathcal{O}))$; and, by the Lemma 2.2, for all $m, n \in \mathbb{N}$, we have

$$\int_0^T \int_{\mathcal{O}} (\|u_n - u_m\|^2 + \|u'_n - u'_m\|^2) dxdt \geq K_2 E_{0,1}(u_{0n} - u_{0m}, u_{1n} - u_{1m}) \quad (2.14)$$

where $K_2 \in \mathbb{R}_+^*$.

So, the Cauchy sequence (u_{0n}, u_{1n}) in $H_0^1(\Omega)$ converges to (u_0, u_1) in $H_0^1(\Omega)$ and u'_n converges to u' in $L^2(0, T; L^2(\mathcal{O}))$: consequently $u' = 0$ in $]0, T[\times \mathcal{O}$.

Furthermore u is constant over $]0, T[\times \mathcal{O}$ and $\gamma u = 0$ over $]0, T[\times \partial\mathcal{O}$: we get now that $u = 0$ in $]0, T[\times \mathcal{O}$, which is absurd according to (2.11) completing the proof \square

Proof. Theorem 2.1

From the previous lemmas 2.2 and 2.3, we deduce that the application

$$(u_0, u_1) \mapsto \left\{ \int_0^T \int_{\mathcal{O}} \|u'\|^2 \right\}^{\frac{1}{2}} \quad (2.15)$$

well defined on $H_0^1(\Omega) \times H^{-1}(\Omega)$ a standard equivalent to that of energy.

Now, let $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ and consider $\psi \in H_0^1(\Omega)$ solution of problem

$$\begin{cases} -\Delta\psi = u_1 \text{ in } \Omega \\ \gamma\psi = 0 \text{ on } \Gamma \end{cases} \quad (2.16)$$

and, assume

$$\phi(t, x) = \int_0^t u(s, x) ds + \psi(x) \quad (2.17)$$

The function ϕ thus defined is solution of the equation of homogeneous waves (1.1) with the initial conditions (ϕ_0, u_0) and checks $\phi' = u$.

From previous estimates, we can deduce that there is a constant $C_2(\mathcal{O}) > 0$ such that

$$\int_0^T \int_{\mathcal{O}} \|u\|^2 dxdt = \int_0^T \int_{\mathcal{O}} \|\phi'\|^2 dxdt \geq C_2(\mathcal{O}) \{ \|\psi\|^2 + \|u_0\|^2 \} \quad (2.18)$$

Knowing that the operator $-\Delta : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ is an isomorphism, we deduce that there is a constant $C_3(\mathcal{O}) > 0$ such that

$$\int_0^T \int_{\mathcal{O}} \|u\|^2 dxdt \geq C_3(\mathcal{O}) \left\{ \|u_1\|_{H^{-1}(\Omega)}^2 + \|u_0\|_{L^2(\Omega)}^2 \right\}. \quad (2.19)$$

\square

3 Result of Exact Regional Controllability in the Neighborhood of a Strategic Border Area

Now with these estimates (called observability's inequalities), we are able to roll out the Lions HUM method in the well chosen strategies open domain \mathcal{O} on the boundary .

Thus, without harming the generality of the work, one is restricted to a domain with only one crack.

Theorem 3.1. (Main result) Let (μ, \mathcal{O}) a strategic area actuator, there exists a time $T_0 > 0$ such that for all $T > T_0$, $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and f a linear profile function, there exists a control $v_\mu \in L^2(\mathcal{O})$ with support in $]0, T[\times \mathcal{O}$ such that if y is solution of equation

$$\begin{cases} y'' - \Delta y = f & \text{in } Q_T \\ \gamma y = \chi_{\mathcal{O}_j^T} v_\mu & \text{on } \mathcal{O} \\ y(0) = y_0 & \text{in } \Omega \\ y'(0) = y_1 & \text{in } \Omega \end{cases} \quad (3.1)$$

then $y(T) = y'(T) = 0$.

Proof. Let $(y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, f a linear second member profile function and $y \in \mathcal{C}(0, T; L^2(\Omega)) \cap \mathcal{C}^1(0, T; H^{-1}(\Omega))$ the solution of equation

$$\begin{cases} y'' - \Delta y = f & \text{in } Q_T \\ \gamma y = \chi_{\mathcal{O}_j^T} v_\mu & \text{on } \mathcal{O} \\ y(0) = y_0 & \text{in } \Omega \\ y'(0) = y_1 & \text{in } \Omega \end{cases}$$

Let now $y \in \mathcal{C}(0, T; H_0^1(\Omega)) \cap \mathcal{C}^1(0, T; L^2(\Omega))$ as the only solution of the homogeneous equation

$$\begin{cases} y'' - \Delta y = 0 & \text{in } Q_T \\ \gamma y = \chi_{\mathcal{O}_j^T} v_\mu & \text{on } \mathcal{O} \\ y(0) = y_0 & \text{in } \Omega \\ y'(0) = y_1 & \text{in } \Omega \end{cases} \quad (3.2)$$

and, we pose Λ the application defined by $\Lambda(y_0, y_1) = (-y'(0), y(0))$.

The application Λ is continous linear and then we have:

$$\langle \Lambda(y_0, y_1), (y_0, y_1) \rangle_{L^2(\Omega) \times H_0^1(\Omega), L^2(\Omega) \times H^{-1}(\Omega)} = \int_0^T \int_\Omega \|y\|^2 dx dt \quad (3.3)$$

More from Theorem 2.1, we deduce that $\Lambda : L^2(\Omega) \times H^{-1}(\Omega) \longrightarrow L^2(\Omega) \times H_0^1(\Omega)$ is a isomorphism which completes the proof. \square

Remark 3.1. 1. This work is carried out on a nonconvex polygonal domain with only one crack to reduce computations. But one obtains the same results on a field with several cracks under the only condition that the lines of cracks are concurrent, thus respecting the geometric conditions decreed by Grisvard [10].

2. Likewise these results remain valid for a domain with corners see Seck [19], [15].

4 Conclusion and Perspective

In this work, we have proposed a problem resulting from numerous problems linked to physics, in particular to electrical engineering: many devices present geometric singularities which produce phenomena undesirable.

- a** Grisvard [8], [9] adapted the Hilbertian Uniqueness Method (HUM) from Lions in openings with polygonal border with corners: we are in the presence of singular solutions. Thus, Grisvard was led to impose geometric restrictions on the open considered to ensure the minimum regularity necessary for the solution of the wave equation. This allowed him to perform integrations by parts by the multiplier method.
- b** In our work, we combined the notion of strategic boundary actuators due to El Jai [13] and the Grisvard [7] multiplier method to establish an exact controllability result without additional geometric conditions. And this allowed the regularization of the solution and to make the integrations by parts necessary to solve the system in these singular domains.
- c** In the near future, we plan to expand these results without any geometric conditions in a few areas with corners or cracks and in higher dimension.

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Competing Interests

Authors have declared that no competing interests exist.

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