Physical Science International Journal

24(3): 61-75, 2020; Article no.PSIJ.57303
ISSN: 2348-0130

Eigen-Solutions to Schrodinger Equation with Trigonometric Inversely Quadratic Plus Coulombic Hyperbolic Potential

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Authors' contributions

This work was carried out in collaboration among all authors. Author IBO designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors ADA and AOA managed the analyses of the study. Author IEE managed the literature searches. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/PSIJ/2020/v24i330183
Editor(s):
(1) Dr. Horacio S. Vieira, University of São Paulo, Brazil.
(2) Dr. Thomas F. George, University of Missouri-St. Louis, USA.
Reviewers:
(1) M. Abu-Shady, Menoufia University, Egypt.
(2) Pasupuleti Venkata Siva Kumar, India.
(3) Ratinan Boonklurb, Chulalongkorn University, Thailand.
Complete Peer review History: http://www.sdiarticle4.com/review-history/57303

Original Research Article

Received 10 March 2020
Accepted 15 May 2020
Published 26 May 2020

ABSTRACT

In this work, we applied parametric Nikiforov-Uvarov method to analytically obtained eigen solutions to Schrodinger wave equation with Trigonometric Inversely Quadratic plus Coulombic Hyperbolic Potential. We obtain energy-Eigen equation and total normalised wave function expressed in terms of Jacobi polynomial. The numerical solutions produce positive and negative bound state energies which signifies that the potential is suitable for describing both particle and anti-particle. The numerical bound state energies decreases with an increase in quantum state with fixed orbital angular quantum number $l = 0$, 1, 2 and 3. The numerical bound state energies decreases with an increase in the screening parameter $\alpha = 0.1, 0.2, 0.3, 0.4$ and 0.5. The energy spectral diagrams show unique quantisation of the different energy levels. This potential reduces to Coulomb potential as a special case. The numerical solutions were carried out with algorithm implemented using MATLAB 8.0 software using the resulting energy-Eigen equation.

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Keywords: Schrodinger equation; Nikiforov-Uvarov method; TIQPCHP.

1. INTRODUCTION

Eigen-solutions to relativistic and nonrelativistic wave equations has been of growing interest for decades because of its applications to some physical systems. The Schrodinger wave equation constitute the nonrelativistic wave equation while Klein-Gordon and Dirac constitutes the relativistic wave equations [1-8]. Most potentials are modelled and applied to solve some physical systems examples include: Morse potential, Tietz-Wei, pseudoharmonic, Deng-Fan, Kratzer –Feus, Mie-Type and many of exponential –type potentials [9-15]. Most of the hyperbolic and trigonometric potentials are applicable in nuclear and high energy physics [16-17]. Most recently, some physical potential has been modelled in trigonometric and hyperbolic potential well. Ikhdair [18] calculated a rotational and vibrational energies of diatomic molecules in Klein-Gordon equation using hyperbolic scalar and vector potentials by means of parametric generalisation of Nikiforov-Uvarov method. Dong et al. [19] examined quantum information entropies for a squared tangent potential where they calculated position Shannon and momentum entropies that satisfies Beckner, Bialynicki –Birula and Mycieslki (BBM) inequality as expected in existing literature. Sun et al. [20] calculated the position and momentum space information entropies using Asymmetric –trigonometric Rosen –Morse potential. Most of the trigonometric and hyperbolic-type potentials belongs to Poschl-Teller family. Recently, Onate [21] examined bound state solutions of the Schrodinger equation with second Poschl-Teller-like potential where he obtained vibrational partition function, mean energy and mean free energy. This Poschl-Teller like potential was expressed in form of hyperbolic cosh and sinh. Majority of trigonometric and hyperbolic potentials are applied in entropic measures to investigate position and momentum space entropies, squeeze state, expectation values and many others. In this work, we calculate analytically the bound state solutions of Schrodinger wave equation using a combined trigonometric and hyperbolic potentials called Trigonometric Inversely Quadratic Plus Coulombic Hyperbolic Potential (TIQPCHP) using Perkeris like approximation to the centrifugal term. This potential does not belong to Poschl-Teller like family due to its combination and that is why it is difficult for the authors to apply it to information entropic measures. The potential is applicable only for a physical system where the bound state energies obtained can be use to study to motion of quarks, mesons, neutrinos and other elementary particles in high energy physics. Shady and Alaraba [22] Studied the trigonometric Rosen-Morse potential using N-radial Schrodinger equation to investigate the interaction between quark and antiquark. They result energy Eigen equation was use to calculate the mass of mesons like: charmonium and bottomonium. Shady et al. [23] calculated the thermodynamic properties of heavy mesons in nonrelativistic quark model using Cornell potential within the framework of Nikiforov-Uvarov method. Their result was applied to calculate mass spectra of charmonium and bottomnium as well as their thermodynamic properties. Shady [24] studied heavy quarkonia and b $\bar{c}$ mesons in the Cornell potential with harmonic oscillator using N-dimensional Schrodinger wave equation where the mass spectra of charmonium and bottomnium were calculated as well. This manuscript is arranged as follows. Parametric Nikiforov-Uvarov method is discussed in Section 2. Section 3 gives the radial solution of the proposed potential using parametric Nikiforov-Uvarov method where energy eigen equation and the total wave functions are obtained alongside with the special case.

We present the numerical calculations of the bound state energies in Section 4 and their corresponding energy spectral diagrams. The analytical calculations of the normalisation constant is presented in Section 5. The results and discussions are presented in section 6 while the article is concluded in Section 7.

The potential model consider in this work is given as:

$$V(r)=\frac{v_0\sin\alpha}{r^2}+\frac{A\cosh\alpha}{r}+B \tag{1}$$

where A and B are real constant and α the screening and adjustable parameter which determines the strength of the potential. The potential plot of equation (1) is given as.

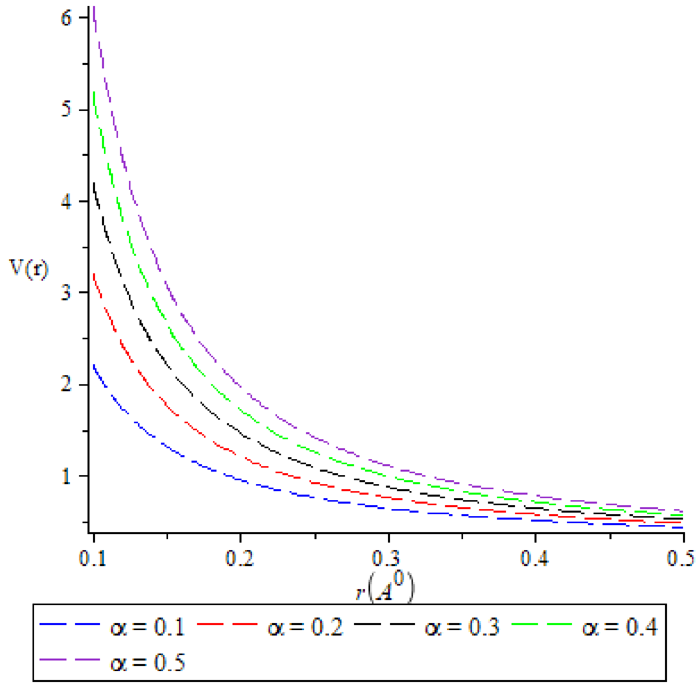

Fig. 1. The potential plot for TIQPCHP

The Pekeris type approximation to the centrifugal term is defined as

$$\frac{1}{r^2} = \frac{\alpha^2}{\left(1-e^{-\alpha r}\right)^2} \Rightarrow \frac{1}{r} = \frac{\alpha}{\left(1-e^{-\alpha r}\right)} \tag{2}$$

The graph of equation (2) for various values of the screening parameter is given below

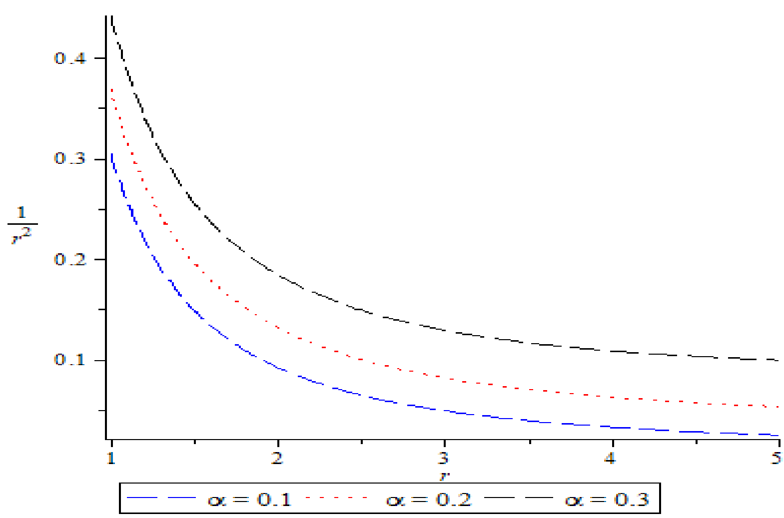

Fig. 2. The Perkeris type approximation

2. BRIEF REVIEW OF PARAMETRIC NIKIFOROV-UVAROV (NU) METHOD

Nikiforov-Uvarov method can either be parametric or conventional. The NU method is based on reducing second order linear differential equation to a generalized equation of hyper-geometric type and provides exact solutions in terms of special orthogonal functions like Jacobi and Laguerre as well as corresponding energy eigen values [25-33]. The parametric NU differential equation is given as

$$\Psi''(s)+\frac{c_1-c_2 s}{s(1-c_3 s)}\Psi'(s)+\frac{1}{s^2(1-c_3 s)^2}\left[-\xi_1 s^2+\xi_2 s-\xi_3\right]\Psi(s)=0 \tag{3}$$

The parametric constants are obtained as follows

$$\begin{bmatrix} c_1=c_2=c_3=1 \\ c_4=\frac{1}{2}\left(1-c_1\right) \\ c_5=\frac{1}{2}(c_2-2c_3) \\ c_6=c_5^2+\xi_1 \\ c_7=2c_4c_5-\xi_2 \\ c_8=c_4^2+\xi c_3 \\ c_9=c_3c_7+c_3^2c_8+c_6 \\ c_{10}=c_1+2c_4+2\sqrt{c_8} \\ c_{11}=c_2-2c_5+2(\sqrt{c_9}+c_3\sqrt{c_8}) \\ c_{12}=c_4+\sqrt{c_8} \\ c_{13}=c_5-\left(\sqrt{c_9}+c_3\sqrt{c_8}\right) \end{bmatrix}. \tag{4}$$

The parametric energy-eigen equation is given as

$$c_2 n-(2n+1)c_5+(2n+1)\left(\sqrt{c_9}+c_3\sqrt{c_8}\right)+n(n-1)c_3+c_7+2c_3c_8+2\sqrt{c_8c_9}=0. \tag{5}$$

The total wave function is given as

$$\Psi(s)=\phi(s)\chi_n(s)=N_n s^{c_{12}}(1-c_3 s)^{c_{13}}P_n^{(c_{10},c_{11})}(1-2c_3 s). \tag{6}$$

3. THE RADIAL SOLUTION OF SCHRODINGER WAVE EQUATION USING THE PROPOSED POTENTIAL

One dimensional Schrodinger wave equation is given as

$$\frac{d^2R(r)}{dr^2}+\frac{2\mu}{\hbar^2}\left[E-V(r)-\frac{\hbar^2 l(l+1)}{2\mu r^2}\right]R(r)=0 \tag{7}$$

Substituting (1) into (7) gives

$$\frac{d^2R(r)}{dr^2}+\frac{2\mu}{\hbar^2}\left[E-\frac{v_0\sin\alpha}{r^2}-\frac{A\cosh\alpha}{r}-B-\frac{\hbar^2 l(l+1)}{2\mu r^2}\right]R(r)=0\,. \tag{8}$$

Substituting (2) into (1) gives

$$\frac{d^2R(r)}{dr^2}+\frac{2\mu}{\hbar^2}\left[E-\frac{v_0\alpha^2\sin\alpha}{\left(1-e^{-\alpha r}\right)^2}-\frac{A\alpha\cosh\alpha}{\left(1-e^{-\alpha r}\right)}-B-\frac{\alpha^2\hbar^2 l(l+1)}{2\mu\left(1-e^{-\alpha r}\right)^2}\right]R(r)=0\,. \tag{9}$$

Let $s=e^{-\alpha r}$ then

$$\frac{d^2s}{dr^2}+\frac{\left(1-s\right)}{s\left(1-s\right)}\frac{ds}{dr}+\frac{1}{s^2}\frac{2\mu}{\hbar^2\alpha^2}\left[E-\frac{v_0\alpha^2\sin\alpha}{\left(1-s\right)^2}-\frac{A\alpha\cosh\alpha}{\left(1-s\right)}-B-\frac{\alpha^2\hbar^2 l(l+1)}{2\mu\left(1-s\right)^2}\right]R(s)=0 \tag{10}$$

Equation (10) can further be reduce to

$$\frac{d^2s}{dr^2}+\frac{\left(1-s\right)}{s\left(1-s\right)}\frac{ds}{dr}+\frac{1}{s^2\left(1-s\right)^2}\left[\begin{matrix}-\left(\varepsilon^2+\chi_2\right)s^2+\left(2\varepsilon^2+\chi_1+2\chi_2\right)s\\-\left(\varepsilon^2+\chi_2+\chi_1+\delta^2+l(l+1)\right)\end{matrix}\right]R(s)=0\,. \tag{11}$$

Where $\varepsilon^2=-\frac{2\mu E_{nl}}{\hbar^2\alpha^2},\quad \delta^2=\frac{2\mu v_0\sin\alpha}{\hbar^2},\chi_1=\frac{2\mu A\cosh\alpha}{\hbar^2\alpha},\chi_2=\frac{2\mu B}{\hbar^2\alpha^2}.$ (12)

Comparing (11) to (3) and by using (4), the following parametric constant can be obtain

$$\xi_1=\left(\varepsilon^2+\chi_2\right),\quad \xi_2=\left(2\varepsilon^2+\chi_1+2\chi_2\right),\quad \xi_3=\left(\varepsilon^2+\chi_2+\chi_1+\delta^2+l(l+1)\right) \tag{13}$$

$$\left[\begin{array}{l}c_1=c_2=c_3=1\\ c_4=0\\ c_5=-\frac{1}{2}\\ c_6=\frac{1}{4}+\varepsilon^2+\chi_2\\ c_7=-2\varepsilon^2-\chi_1-2\chi_2\\ c_8=\varepsilon^2+\chi_2+\chi_1+\delta^2+l(l+1)\\ c_9=\frac{1}{4}+\delta^2+l(l+1)\\ c_{10}=1+2\sqrt{\varepsilon^2+\chi_2+\chi_1+\delta^2+l(l+1)}\\ c_{11}=2+\sqrt{4\delta^2+4l(l+1)+1}+2\sqrt{\varepsilon^2+\chi_2+\chi_1+\delta^2+l(l+1)}\\ c_{12}=\sqrt{\varepsilon^2+\chi_2+\chi_1+\delta^2+l(l+1)}\\ c_{13}=-\frac{1}{2}-\left[\frac{1}{2}\sqrt{4\delta^2+4l(l+1)+1}+\sqrt{\varepsilon^2+\chi_2+\chi_1+\delta^2+l(l+1)}\right]\end{array}\right]. \tag{14}$$

The energy eigen equation can be calculated by substituting equations (13) and (14) into (5) gives

$$\varepsilon^2 = \left(\frac{\left(n^2+n+\frac{1}{2}\right)+\left(n+\frac{1}{2}\right)\sqrt{1+4\delta^2+4l(l+1)}+\chi_1+2\delta^2+2l(l+1)}{(2n+1)+\sqrt{1+4\delta^2+4l(l+1)}} \right)^2 - \chi_2-\chi_1-\delta^2-l(l+1). \quad (15)$$

Substituting (12) into (15) gives the energy eigen equation as

$$E_{nl} = -\frac{\hbar^2\alpha^2}{2\mu}\left(\frac{\left(n^2+n+\frac{1}{2}\right)+\left(n+\frac{1}{2}\right)\sqrt{1+\frac{8\mu v_0\sin\alpha}{\hbar^2}+4l(l+1)}+\frac{2\mu A\cosh\alpha}{\hbar^2\alpha}+\frac{4\mu v_0\sin\alpha}{\hbar^2}+2l(l+1)}{(2n+1)+\sqrt{1+\frac{8\mu v_0\sin\alpha}{\hbar^2}+4l(l+1)}} \right)^2 . \quad (16)$$

$$+B+A\alpha\cosh\alpha+\alpha^2 v_0\sin\alpha+\frac{\hbar^2\alpha^2 l(l+1)}{2\mu}$$

Using (6), the total wave function for the proposed potential is given by

$$\Psi_{nl} = N_{nl} s^{\sqrt{-\frac{2\mu E_{nl}}{\hbar^2\alpha^2}+\frac{2\mu B}{\hbar^2\alpha^2}+\frac{2\mu A\cosh\alpha}{\hbar^2\alpha}+\frac{2\mu v_0\sin\alpha}{\hbar^2}+l(l+1)}} (1-s)^{-\frac{1}{2}-\left[\frac{\frac{1}{2}\sqrt{1+\frac{8\mu v_0\sin\alpha}{\hbar^2}+4l(l+1)}+}{\sqrt{-\frac{2\mu E_{nl}}{\hbar^2\alpha^2}+\frac{2\mu B}{\hbar^2\alpha^2}+\frac{2\mu A\cosh\alpha}{\hbar^2\alpha}+\frac{2\mu v_0\sin\alpha}{\hbar^2}+l(l+1)}}\right]} \times$$

$$P_n^{\left[\left(1+2\sqrt{-\frac{2\mu E_{nl}}{\hbar^2\alpha^2}+\frac{2\mu B}{\hbar^2\alpha^2}+\frac{2\mu A\cosh\alpha}{\hbar^2\alpha}+\frac{2\mu v_0\sin\alpha}{\hbar^2}+l(l+1)}\right),\left(2+\sqrt{1+\frac{8\mu v_0\sin\alpha}{\hbar^2}+4l(l+1)}+2\sqrt{-\frac{2\mu E_{nl}}{\hbar^2\alpha^2}+\frac{2\mu B}{\hbar^2\alpha^2}+\frac{2\mu A\cosh\alpha}{\hbar^2\alpha}+\frac{2\mu v_0\sin\alpha}{\hbar^2}+l(l+1)}\right)\right]}(1-2s) \quad (17)$$

3.1 Special Case

Coulomb potential: Substituting $A=1, \alpha=B=0$ into equation (1), the potential reduces to Coulomb potential $V(r)=\frac{A}{r}$ and the corresponding eigen energy equation is

$$E_{nl} = -\frac{\hbar^2\alpha^2}{2\mu}\left(\frac{\left(n^2+n+\frac{1}{2}\right)+\left(n+\frac{1}{2}\right)\sqrt{1+4l(l+1)}+2l(l+1)}{(2n+1)+\sqrt{1++4l(l+1)}} \right)^2 + \frac{\hbar^2\alpha^2 l(l+1)}{2\mu}. \quad (18)$$

4. NUMERICAL BOUND STATE ENERGIES FOR THE PROPOSE POTENTIAL

We implemented MATLAB algorithm using equation (16) with different orbital angular quantum number. We adopted the following fixed real constant parameters $v_0=A=0.1, B=0.2, \quad \hbar=\mu=1$, with adjustable screening parameter varying from $\alpha=0.1$ to 0.5

Table 1. Numerical bound state energy for Schrodinger equation for $\alpha = 0.1$

n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$
0	**0**	0.1988204727	**0**	**1**	0.1999999965	**0**	**2**	0.1998631421	**0**	**3**	0.1996901744
1	**0**	0.1987136453	**1**	**1**	0.1965293766	**1**	**2**	0.1950075124	**1**	**3**	0.1939584923
2	**0**	0.1931126507	**2**	**1**	0.1887438717	**2**	**2**	0.1855561554	**2**	**3**	0.1832039211
3	**0**	0.1845598182	**3**	**1**	0.1779319895	**3**	**2**	0.1727791340	**3**	**3**	0.1687576855
4	**0**	0.1733752330	**4**	**1**	0.1644126026	**4**	**2**	0.1571120530	**4**	**3**	0.1511761068
5	**0**	0.1596385566	**5**	**1**	0.1482951488	**5**	**2**	0.1387369852	**5**	**3**	0.1307250413
6	**0**	0.1433772158	**6**	**1**	0.1296253900	**6**	**2**	0.1177409515	**6**	**3**	0.1075445468
7	**0**	0.1246027028	**7**	**1**	0.1084252044	**7**	**2**	0.0941698004	**7**	**3**	0.0817140994
8	**0**	0.1033205198	**8**	**1**	0.0847061215	**8**	**2**	0.0680495506	**8**	**3**	0.0532815201

Table 2. Numerical bound state energy for Schrodinger equation for $\alpha = 0.2$

n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$
0	**0**	0.1999984632	**0**	**1**	0.1987832390	**0**	**2**	0.1978162235	**0**	**3**	0.1972223087
1	**0**	0.1884735507	**1**	**1**	0.1800456405	**1**	**2**	0.1747666840	**1**	**3**	0.1712798400
2	**0**	0.1638342365	**2**	**1**	0.1471631221	**2**	**2**	0.1352643859	**2**	**3**	0.1266136009
3	**0**	0.1287155386	**3**	**1**	0.1030751656	**3**	**2**	0.0832175806	**3**	**3**	0.0678250792
4	**0**	0.8345598137	**4**	**1**	0.4851130356	**4**	**2**	0.0199680137	**4**	**3**	-0.0031618892
5	**0**	0.2814049825	**5**	**1**	-0.1627768503	**5**	**2**	-0.0539231424	**5**	**3**	-0.0854276430
6	**0**	-0.3720153621	**6**	**1**	-0.9118682655	**6**	**2**	-0.1381875593	**6**	**3**	-0.1784875761
7	**0**	-0.1125577772	**7**	**1**	-0.1761658974	**7**	**2**	-0.2326838163	**7**	**3**	-0.2820666437
8	**0**	-0.1979223018	**8**	**1**	-0.2711884160	**8**	**2**	-0.3373316374	**8**	**3**	-0.3959993274

Table 3. Numerical bound state energy for Schrodinger equation for $\alpha = 0.3$

n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$
0	**0**	0.1986991473	**0**	**1**	0.19517448620	**0**	**2**	0.1933538980	**0**	**3**	0.1923217286
1	**0**	0.1679348678	**1**	**1**	0.15006268400	**1**	**2**	0.1390423109	**1**	**3**	0.1318348023
2	**0**	0.1112581315	**2**	**1**	0.07495527010	**2**	**2**	0.0489886205	**2**	**3**	0.0301646895
3	**0**	0.0315464186	**3**	**1**	-0.02483780344	**3**	**2**	-0.0687913845	**3**	**3**	-0.1028374075

n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$
4	**0**	-0.0708236177	**4**	**1**	-0.14799502020	**4**	**2**	-0.2115431940	**4**	**3**	-0.2630509874
5	**0**	-0.1957567623	**5**	**1**	-0.29406139620	**5**	**2**	-0.3781140516	**5**	**3**	-0.4485056936
6	**0**	-0.3432199437	**6**	**1**	-0.46284628620	**6**	**2**	-0.5679525195	**6**	**3**	-0.6581630196
7	**0**	-0.5131992250	**7**	**1**	-0.65425841230	**7**	**2**	-0.7807678720	**7**	**3**	-0.8914334487
8	**0**	-0.7056879062	**8**	**1**	-0.86824962030	**8**	**2**	-1.0163950400	**8**	**3**	-1.1479621610

Table 4. Numerical bound state energy for Schrodinger equation for $\alpha = 0.4$

n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$
0	**0**	0.1950785721	**0**	**1**	0.1892493492	**0**	**2**	0.1865274182	**0**	**3**	0.1850271879
1	**0**	0.1370418660	**1**	**1**	0.1066569946	**1**	**2**	0.0879157225	**1**	**3**	0.0756971120
2	**0**	0.0351078134	**2**	**1**	-0.0278558252	**2**	**2**	-0.0731998599	**2**	**3**	-0.1060631653
3	**0**	-0.1074476165	**3**	**1**	-0.2058533884	**3**	**2**	-0.2832046614	**3**	**3**	-0.3431599046
4	**0**	-0.2901881662	**4**	**1**	-0.4252306301	**4**	**2**	-0.5374159217	**4**	**3**	-0.6284401940
5	**0**	-0.5130027767	**5**	**1**	-0.6852617097	**5**	**2**	-0.8338727521	**5**	**3**	-0.9584825974
6	**0**	-0.7758527203	**6**	**1**	-0.9856421918	**6**	**2**	-1.1716367390	**6**	**3**	-1.3314833890
7	**0**	-1.0787216340	**7**	**1**	-1.3262262170	**7**	**2**	-1.5502129830	**7**	**3**	-1.7464184890
8	**0**	-1.4216016360	**8**	**1**	-1.7069367970	**8**	**2**	-1.9693204180	**8**	**3**	-2.2026714360

Table 5. Numerical bound state energy for Schrodinger equation for $\alpha = 0.5$

n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$
0	**0**	0.1892880088	**0**	**1**	0.1810855670	**0**	**2**	0.1773899226	**0**	**3**	0.175379002
1	**0**	0.0957818770	**1**	**1**	0.0499155408	**1**	**2**	0.0214744600	**1**	**3**	0.002945097
2	**0**	-0.0648119310	**2**	**1**	-0.1612265357	**2**	**2**	-0.2312188139	**2**	**3**	-0.281982601
3	**0**	-0.2886485251	**3**	**1**	-0.4399886032	**3**	**2**	-0.5599630539	**3**	**3**	-0.653061466
4	**0**	-0.5752077683	**4**	**1**	-0.7832804604	**4**	**2**	-0.9576230433	**4**	**3**	-1.099263876
5	**0**	-0.9243563897	**5**	**1**	-1.1900350610	**5**	**2**	-1.4212092400	**5**	**3**	-1.615313492
6	**0**	-1.3360477450	**6**	**1**	-1.6598044440	**6**	**2**	-1.9492903790	**6**	**3**	-2.198428115
7	**0**	-1.8102620790	**7**	**1**	-2.1923738420	**7**	**2**	-2.5411114120	**7**	**3**	-2.847027866
8	**0**	-2.3469898600	**8**	**1**	-2.7876298390	**8**	**2**	-3.1962434310	**8**	**3**	-3.560161581

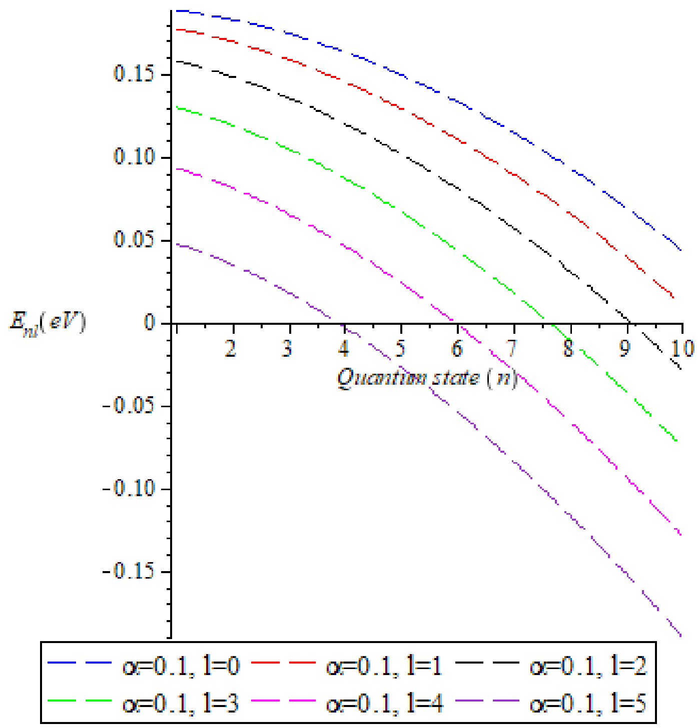

Fig. 3. Bound state energy spectral diagram for $\alpha = 0.1$

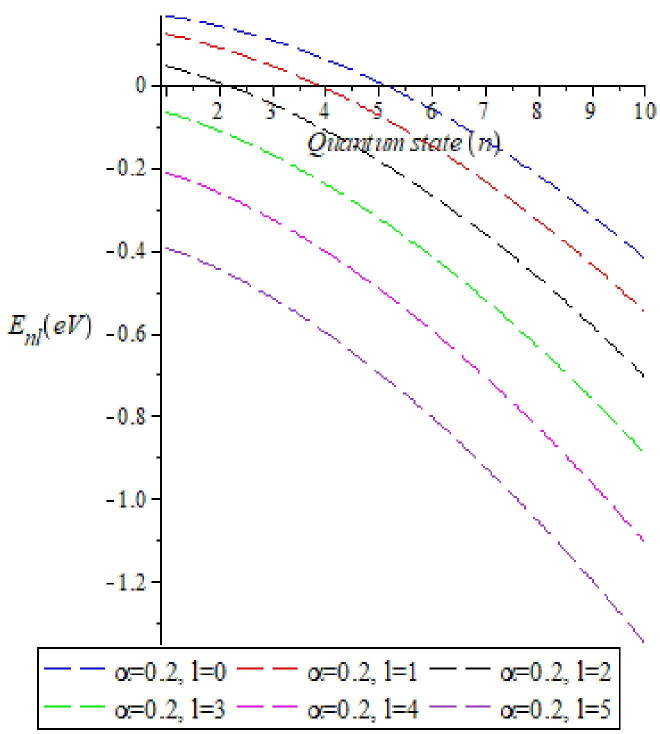

Fig. 4. Bound state energy spectral diagram for $\alpha = 0.2$

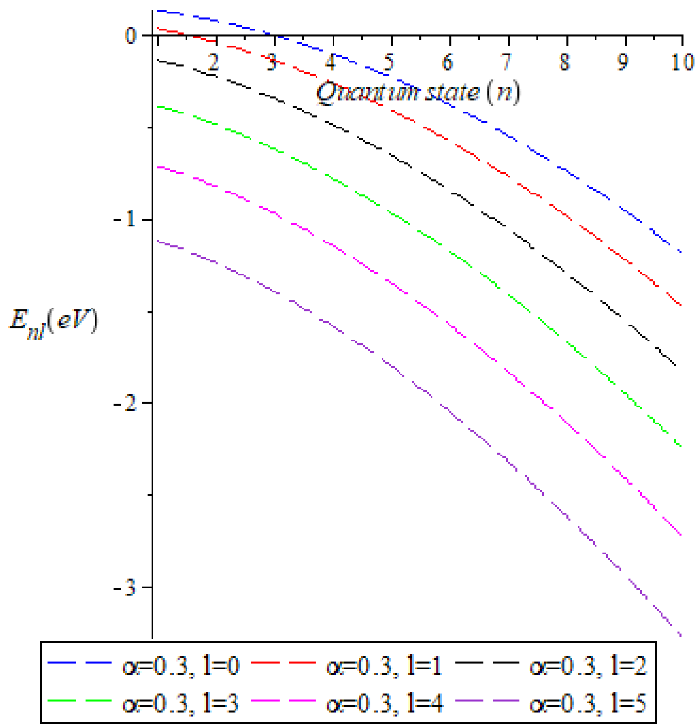

Fig. 5. Bound state energy spectral diagram for $\alpha = 0.3$

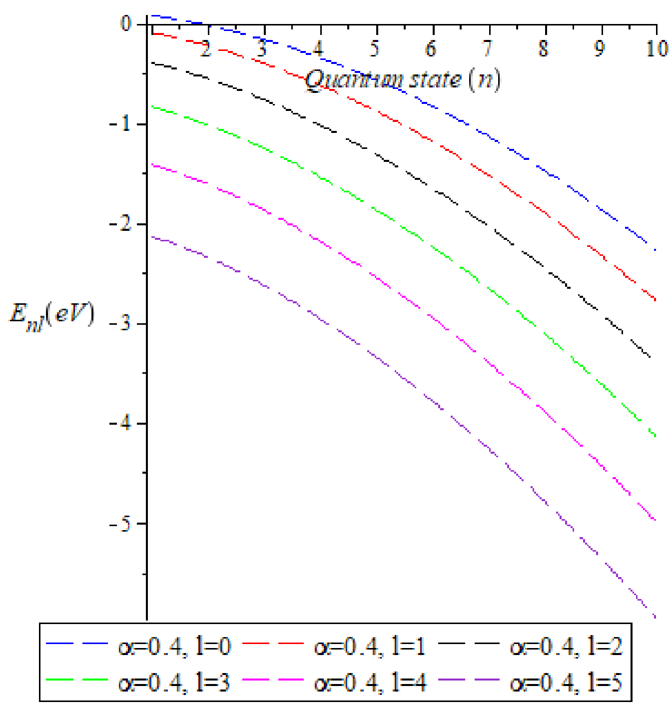

Fig. 6. Bound state energy spectral diagram for $\alpha = 0.4$

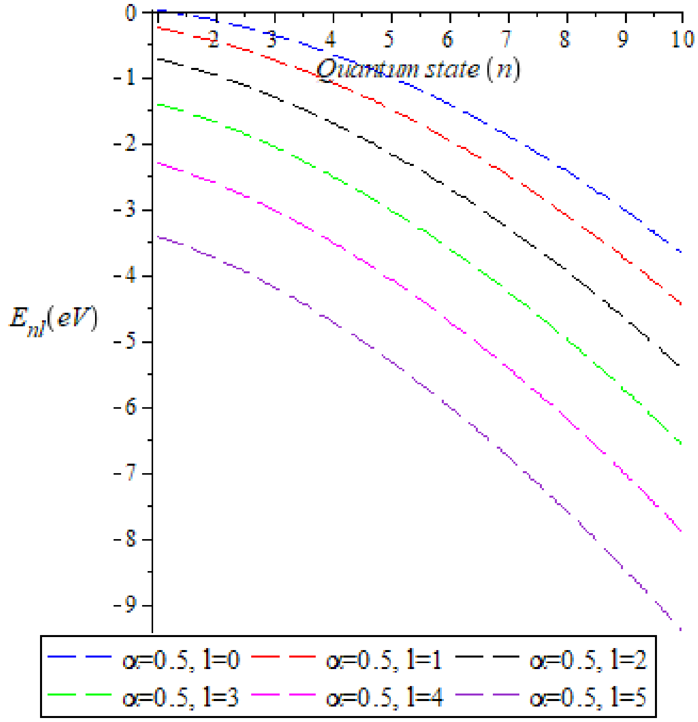

Fig. 7. Bound state energy spectral diagram for $\alpha = 0.5$

5. ANALYTICAL CALCULATION OF NORMALIZATION CONSTANT

The total wave function is given in equation (17). Considering equation (17), let

$$M_1 = \sqrt{-\frac{2\mu E_{nl}}{\hbar^2\alpha^2} + \frac{2\mu B}{\hbar^2\alpha^2} + \frac{2\mu A\cosh\alpha}{\hbar^2\alpha} + \frac{2\mu v_0 \sin\alpha}{\hbar^2} + l(l+1)}$$

$$M_2 = \sqrt{1 + \frac{2\mu v_0 \sin\alpha}{\hbar^2} + 4l(l+1)} .$$

Equation (17) simplify to

$$\Psi_{nl}(s) = B_{nl} S^{M_1} (1-s)^{-\frac{1}{2}-\left[\frac{1}{2}M_2+M_1\right]} P_n^{[(1+2M_1),(2+M_2+2M_1)]}(1-2s) . \tag{19}$$

Jacobi polynomial can be express in the form

$$P_n^{[\alpha,\beta]}(1-2s) = \frac{\Gamma(n+1+\alpha)}{n!\Gamma(1+\alpha)} {}_2F_1(-n, n+\alpha+\beta+1, \alpha+1; s) . \tag{20}$$

The Jacobi polynomial of equation (17) can then be expressed as

$$P_n^{[(1+2M_1),(2+M_2+2M_1)]}(1-2s) = \frac{\Gamma(n+2+2M_1)}{n!(2+2M_1)} {}_2F_1(-n, n+4+4M_1+M_2, 2+2M_1; s) . \tag{21}$$

The wave function express in hypergeometric polynomial is given as

$$\Psi_{nl}(s) = N_{nl} S^{M_1} (1-s)^{-\frac{1}{2}-\left[\frac{1}{2}M_2+M_1\right]} \frac{\Gamma(n+2+2M_1)}{n!(2+2M_1)} {}_2F_1(-n, n+4+4M_1+M_2, 2+2M_1; s) \quad (22)$$

To normalize a wave function then

$$\int_0^\infty \Psi(r)\Psi^*(r)dr \Rightarrow \int_0^\infty |\Psi(r)|^2 dr = 1. \quad (23)$$

Substituting equation (22) into (23) and recalling that $s = e^{-\alpha r}$ gives

$$\begin{aligned}
&\int_0^\infty |\Psi(r)|^2 dr = 1 \Rightarrow \frac{N_{nl}^2}{\alpha} \int_1^0 s^{2M_1} (1-s)^{-2\left(\frac{1}{2}+\frac{M_2}{2}+M_1\right)} \left[P_n^{[(1+2M_1),(2+2M_1+M_2)]}(1-2s)\right]^2 ds = 1 \\
&\frac{N_{nl}^2}{\alpha} \int_1^0 s^{2M_1} (1-s)^{-2\left(\frac{1}{2}+\frac{M_2}{2}+M_1\right)} \left[P_n^{\left[(1+2M_1),2\left(\frac{1}{2}+M_1+\frac{M_2}{2}+\frac{1}{2}\right)\right]}(1-2s)\right]^2 \frac{ds}{s} = 1 \\
&M_3 = \left(\frac{1}{2}+\frac{M_2}{2}+M_1\right) \\
&\frac{N_{nl}^2}{\alpha} \int_1^0 s^{2M_1-1} (1-s)^{-2M_3} \left[P_n^{[(1+2M_1),(2M_3+1)]}(1-2s)\right]^2 ds = 1
\end{aligned} \quad (24)$$

Considering the transformation $z = 1-2s$, then the boundary of integration changes from $(0,1) \to (1,-1)$ from s to z coordinate respectively., equation (24) can be written as

$$\frac{N_{nl}^2}{2\alpha} \int_{-1}^1 \left(\frac{1-z}{2}\right)^{2M_1-1} \left(\frac{1+z}{2}\right)^{-2M_3} \left[P_n^{[(1+2M_1),(2M_3+1)]}(z)\right]^2 dz = 1. \quad (25)$$

Considering the standard integral

$$\int_{-1}^1 \left(\frac{1-t}{2}\right)^a \left(\frac{1-t}{2}\right)^{-b} \left[P_n^{(a,b+1)}(t)\right]^2 dt = \frac{2^{a+b+1}\Gamma(a+n+1)\Gamma(b+n+1)}{n!\Gamma(a+b+n+1)\Gamma(a+b+2n+1)} = 1. \quad (26)$$

Expressing equation (25) in terms of (26) gives

$$\frac{N_{nl}^2}{2\alpha} \frac{2^{2M_1+2M_3}\Gamma(2M_1+n)\Gamma(n+2M_3+1)}{n!\Gamma(2M_1+2M_3+n)\Gamma(2M_1+2M_3+2n)} = 1. \quad (27)$$

Therefore, the normalization constant

$$N_{nl} = \sqrt{\frac{2\alpha n!\Gamma(2M_1+2M_3+n)\Gamma(2M_1+2M_3+2n)}{2^{2M_1+2M_3}\Gamma(2M_1+n)\Gamma(n+2M_3+1)}}. \quad (28)$$

Substituting equation (28) into (22) gives the complete total wave function as

$$\Psi_{nl}(s)=\sqrt{\frac{2\alpha n!\Gamma(2M_1+2M_3+n)\Gamma(2M_1+2M_3+2n)}{2^{2M_1+2M_3}\Gamma(2M_1+n)\Gamma(n+2M_3+1)}}S^{M_1}(1-s)^{-\frac{1}{2}-\left[\frac{1}{2}M_2+M_1\right]}\times \frac{\Gamma(n+2+2M_1)}{n!\Gamma(2+2M_1)}\,{}_2F_1(-n,n+4+4M_1+M_2,2+2M_1;s) \qquad (29)$$

6. RESULTS AND DISCUSSION

The numerical results obtain in Tables 2 to 5 has both positive and negative bound state values which shows that the potential is suitable for describing both particle and anti-particle as applied to elementary particles in High energy physics. Table 1 has positive bound state values which is useful in describing a particle like neutrino. The numerical tables show that the bound state energies decrease with an increase in quantum state and also decreases with an increase in the adjustable screening parameter. The numerical bound state spectral diagrams as shown in Figs. 3 to 7 show unique quantization of different energy level which is in consonance to the report in existing literature [34]. The propose exhibit both attractive or repulsive property because it is a combination of both short and long range potential. The developed potential is very applicable in particle and high energy physics. This potential also reduces to a well -known Coulomb potential.

7. CONCLUSION

In this work, we have applied the concept of parametric NU method to analytically calculate the bound state solutions of Schrodinger wave equation using TIQPCHP. The numerical calculations were carried out for different quantum state which shows unique quantization of different energy level. We also obtained the energy-Eigen equation and normalized wave function expressed in hypergeometric Jacobi polynomial. The numerical result also decreases with an increase in quantum state. The study of trigonometric and hyperbolic potential has application in high energy physics.

ACKNOWLEDGEMENT

The authors acknowledge the contributions and kind response of the reviewers which has significantly help in improving the article.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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DOI: 10.14810/ijrap.2015.4403

Peer-review history:
The peer review history for this paper can be accessed here:
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