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Cyclic-antimagic construction of ladders

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Abstract: A simple graph $G = (V, E)$ admits an *H-covering* if every edge in the edge set $E(G)$ belongs to at least one subgraph of G isomorphic to a given graph H. A graph G having an H-covering is called (a,d)-*H-antimagic* if the function $h : V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)| + |E(G)|\}$ defines a bijective map such that, for all subgraphs H' of G isomorphic to H, the sums of labels of all vertices and edges belonging to H' constitute an arithmetic progression with the initial term a and the common difference d. Such a graph is named as *super* (a,d)-H-antimagic if $h(V(G)) = \{1, 2, 3, \ldots, |V(G)|\}$. For $d = 0$, the *super* (a,d)-H-antimagic graph is called *H-supermagic*. In the present paper, we study the existence of *super* (a,d)*-cycle-antimagic* labelings of ladder graphs for certain differences d.

Keywords: Cycle-antimagic, super cycle-antimagic, super (a,d)-cycle-antimagic, C_4-antimagic, ladder graph.

1. Introduction

Let $G = (V, E)$ be a finite simple graph. A family of subgraphs $H_1, H_2, \ldots, H_t$ of the graph G with property that each element of E belongs to at least one subgraph H_i, $i = 1, 2, \ldots, t$, is classified as an *edge-covering* of G. With the possibility, H_i isomorphic to a given graph H, G is said to admit an *H-covering*.

Suppose a (p,q)-graph G admits an H-covering. A bijective function $h : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ is called a total labeling for G. The associated *H-weight* is defined as

$$wt_h(H) = \sum_{v \in V(H)} h(v) + \sum_{e \in E(H)} h(e).$$

A total labeling $h : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ is then called an H-magic labeling, if there exists a positive integer m_c (called the *magic constant*) such that for every subgraph H' of G isomorphic to H, $wt_h(H) = m_c$. The graph G admitting such a labeling is called H-magic.

In addition, if the H-weights constitute an arithmetic progression $a, a+d, a+2d, \ldots, a+(t-1)d$, where $a > 0$ and $d \geq 0$ are two integers, and t is the number of all subgraphs of G isomorphic to H, then we say that graph G is an (a,d)-*H-antimagic*. The restriction $h(V) = \{1, 2, \ldots, p\}$ makes G a *super* (a,d)-*H-antimagic*. If the subgraph H is isomorphic to a cycle C_k for some k, then super (a,d)-H-antimagic labeling is referred to a super (a,d)-cycle-antimagic labeling.

The H-supermagic labelings were first studied by Gutiérrez *et al.* in [1] as an extension of the edge-magic and super edge-magic labelings introduced by Kotzig *et al.* [2] and Enomoto *et al.* [3], respectively. Gutiérrez *et al.* considered *star*-supermagic and *path*-supermagic labelings of some connected graphs and proved that the path P_n and the cycle C_n are P_m-supermagic for some m. Maryati *et al.* [4] gave P_m-supermagic labelings of some trees such as shrubs, subdivision of shrubs and banana tree graphs. Lladó *et al.* [5] investigated C_n-supermagic graphs and proved that wheels, windmills, books and prisms are C_m-magic for some m. Some results on C_n-supermagic labelings of several classes of graphs can be found in [6,7]. Other examples of H-supermagic graphs with different choices of H have been given by Jeyanthi *et al.* in [8]. Inayah, Lladó and Moragas [9] gave a connection between graceful trees and antimagic H-decomposition of complete graphs. Maryati *et al.* [6] investigated the G-supermagicness of a disjoint union of c copies of a graph G and showed that disjoint union of any paths is cP_m-supermagic for some c and m.

Motivated by H-(super)magic labelings, Inayah *et al.* [10] introduced the (a,d)-H-antimagic labeling. In [11] they investigated the super (a,d)-H-antimagic labelings for some shackles of a connected graph H. In

[12] Miller *et al.* exhibit an operation on graphs which keeps super H-antimagic properties using technique of partitioning sets of integers. The existence of super (a,d)-H-antimagic labelings for disconnected graphs is studied in [13] and there is proved that if a graph G admits a (super) (a,d)-H-antimagic labeling, where $d = |E(H)| - |V(H)|$, then the disjoint union of m copies of the graph G, denoted by mG, admits a (super) (b,d)-H-antimagic labeling as well.

The (super) (a,d)-H-antimagic labeling is related to a super d-antimagic labeling of type $(1,1,0)$ of a plane graph that is the generalization of a face-magic labeling introduced by Lih [14]. Further information on super d-antimagic labelings can be found in [15,16].

For $H \cong K_2$, (super) (a,d)-H-antimagic labelings are also called (super) (a,d)-edge-antimagic total labelings and have been introduced in [17]. More results on (a,d)-edge-antimagic total labelings, can be found in [15,18–20]. The vertex version of these labelings for generalized pyramid graphs is given in [21].

A *ladder* is a Cartesian product $L_m \cong P_m \times P_2$ of the path on m vertices with the path on two vertices. The vertex set $V(L_m)$ consists of the elements $\{u_i, v_i : 1 \le i \le m\}$ and the edge set $E(L_m)$ consists of the elements $\{u_iv_i : 1 \le i \le m\} \cup \{u_iu_{i+1}, v_iv_{i+1} : 1 \le i \le m-1\}$.

In [7], Ngurah *et al.* proved that ladder graph L_n is C_4-supermagic for every $n \ge 2$.

In the present paper, we will study the existence of the super cycle-antimagic labelings of ladder graphs L_m. More explicitly, we will describe super (a,d)-C_4-antimagic labelings of ladder graphs for differences $0 \le d \le 15$.

2. Results

Let $C_4^{(i)}, 1 \le i \le m-1$ be the subcycle of L_m with $V(C_4^{(i)}) = \{u_i, u_{i+1}, v_i, v_{i+1}\}$ and $E(C_4^{(i)}) = \{u_iu_{i+1}, v_iv_{i+1}, u_iv_i, u_{i+1}v_{i+1}\}$.
For the C_4-weight of the cycle $C_4^{(i)}$, $i = 1,2,\dots,m-1$, under the total labeling h we get:

$$\begin{aligned} wt_h(C_4^{(i)}) &= h(u_i) + h(v_i) + h(u_{i+1}) + h(v_{i+1}) \\ &\quad + h(u_iu_{i+1}) + h(v_iv_{i+1}) + h(u_iv_i) + h(u_{i+1}v_{i+1}). \end{aligned}$$

The following theorem shows that ladder graph L_m admits super (a,d)-C_4-antimagic labelings for differences $0 \le d \le 6$.

Theorem 1. *Let $m \ge 3$ be a positive integer. Then the ladder L_m admits a super (a,d)-C_4-antimagic labeling for $d \in \{0,1,2,3,4,5,6\}$.*

Proof. Label the vertices of the ladder L_m by the following integers:

$$\begin{aligned} h(u_i) &= i && \text{if } i = 1,2,\dots,m, \\ h(v_i) &= 2m+1-i && \text{if } i = 1,2,\dots,m. \end{aligned}$$

Clearly, under the vertex labeling h the vertices of L_m receive labels from 1 up to $2m$ and for partial weights of $C_4^{(i)}$ for every $i = 1,2,\dots,m-1$ we get

$$w_h = h(u_i) + h(u_{i+1}) + h(v_i) + h(v_{i+1}) = 4m+2. \tag{1}$$

We define the labelings h_j, $j = 0,1,2,\dots,6$, for the edges of L_m in the following way:
if $i = 1,2,\dots,m-1$ then

$$h_j(u_iu_{i+1}) = \begin{cases} 3m-i & \text{for } j = 0, \\ 4m-i & \text{for } j = 1, \\ 3m+i & \text{for } j = 2,3,4, \\ 3m-1+2i & \text{for } j = 5,6, \end{cases}$$

$$h_j(v_iv_{i+1}) = \begin{cases} 4m-1-i & \text{for } j=0, \\ 4m-1+i & \text{for } j=1,3,4, \\ 4m+\lceil \frac{m}{2} \rceil +1-i & \text{for } j=2, \\ 3m+2i & \text{for } j=5,6, \end{cases}$$

and if $i = 1, 2, \ldots, m$ then

$$h_j(u_iv_i) = \begin{cases} 4m-2+i & \text{for } j=0, \\ 2m+\frac{i+1}{2} & \text{for } j=1,3,5 \text{ and } i \text{ odd}, \\ 2m+\lceil \frac{m}{2} \rceil + \frac{i}{2} & \text{for } j=1,3,5 \text{ and } i \text{ even}, \\ 2m+i & \text{for } j=2,4,6. \end{cases}$$

It is not difficult to see that every edge labeling h_j, $j = 0, 1, 2, \ldots, 6$, admits the values from $2m+1$ up to $5m-2$. Thus every edge labeling h_j, $j = 0, 1, 2, \ldots, 6$, together with vertex labeling h has properties of a total labeling of the ladder L_m. Let

$$w_{h_j} = h_j(u_iu_{i+1}) + h_j(v_iv_{i+1}) + h_j(u_iv_i) + h_j(u_{i+1}v_{i+1}) \tag{2}$$

be partial weights of cycles $C_4^{(i)}$ for every $i = 1, 2, \ldots, m-1$ and $j = 0, 1, \ldots, 6$. Then according to (1) and (2) we get

$$wt_{h_j}(C_4^{(i)}) = w_h + w_{h_j} = 4m + 2 + w_{h_j}. \tag{3}$$

For $j = 0$, $w_{h_0} = 15m - 4$ and $wt_{h_0}(C_4^{(i)}) = 19m - 2$ for every $i = 1, 2, \ldots, m-1$. Thus the total labeling $h \cup h_0$ is C_4-supermagic for L_m.

For $j = 1$, $w_{h_1} = 12m + \lceil \frac{m}{2} \rceil + i$ and $wt_{h_1}(C_4^{(i)}) = 16m + \lceil \frac{m}{2} \rceil + 2 + i$ for every $i = 1, 2, \ldots, m-1$. Under the labeling $h \cup h_1$ the C_4-weights of L_m are consecutive integers and L_m is a super $(16m + \lceil \frac{m}{2} \rceil + 3, 1)$-$C_4$-antimagic.

For $j = 2$, $w_{h_2} = 11m + \lceil \frac{m}{2} \rceil + 2 + 2i$ and $wt_{h_2}(C_4^{(i)}) = 15m + \lceil \frac{m}{2} \rceil + 4 + 2i$ for every $i = 1, 2, \ldots, m-1$ and it proves that the total labeling $h \cup h_2$ is a super $(15m + \lceil \frac{m}{2} \rceil + 6, 2)$-$C_4$-antimagic.

For $j = 3$, $w_{h_3} = 11m + \lceil \frac{m}{2} \rceil + 3i$ and $wt_{h_3}(C_4^{(i)}) = 15m + \lceil \frac{m}{2} \rceil + 2 + 3i$ for every $i = 1, 2, \ldots, m-1$. It shows that the labeling $h \cup h_3$ is a super C_4-antimagic with the difference $d = 3$.

For $j = 4$, $w_{h_4} = 11m + 4i$ and $wt_{h_4}(C_4^{(i)}) = 15m + 2 + 4i$ for every $i = 1, 2, \ldots, m-1$. Under the labeling $h \cup h_4$ the C_4-weights of L_m form the arithmetic sequence with the difference $d = 4$ and L_m is a super $(15m + 6, 4)$-C_4-antimagic.

For $j = 5$, $w_{h_5} = 10m + \lceil \frac{m}{2} \rceil + 5i$ and $wt_{h_5}(C_4^{(i)}) = 14m + \lceil \frac{m}{2} \rceil + 2 + 5i$ for every $i = 1, 2, \ldots, m-1$. It shows that the labeling $h \cup h_5$ is a super C_4-antimagic with the difference $d = 5$.

For $j = 6$, $w_{h_6} = 10m + 6i$ and $wt_{h_6}(C_4^{(i)}) = 14m + 2 + 6i$ for every $i = 1, 2, \ldots, m-1$ and it proves that the total labeling $h \cup h_6$ is a super $(14m + 8, 2)$-C_4-antimagic. This completes the proof. □

Next theorem proves a super C_4-antimagicness for ladder graph with differences $7 \le d \le 15$.

Theorem 2. *Let $m \ge 3$ be a positive integer. Then the ladder L_m is a super (a, d)-C_4-antimagic for every $d \in \{7, 8, 9, 10, 11, 12, 13, 14, 15\}$.*

Proof. Define a vertex labeling $h : V(L_m) \to \{1, 2, \ldots, 2m\}$ such that

$$\begin{aligned} h(u_i) &= 2i - 1 && \text{if } i = 1, 2, \ldots, m, \\ h(v_i) &= 2i && \text{if } i = 1, 2, \ldots, m. \end{aligned}$$

For partial weights of $C_4^{(i)}$ for every $i = 1, 2, \ldots, m-1$ we get

$$w_h = h(u_i) + h(u_{i+1}) + h(v_i) + h(v_{i+1}) = 8i + 2. \tag{4}$$

We construct the labelings h_j, $j = 7, 8, \ldots, 15$, for the edges of L_m as follows:
if $i = 1, 2, \ldots, m-1$ then

$$h_j(u_iu_{i+1}) = \begin{cases} 4m - i & \text{for } j = 7, 9, \\ 3m + i & \text{for } j = 8, 11, 12, \\ 2m + 2i - 1 & \text{for } j = 10, 14, \\ 3m - 1 + 2i & \text{for } j = 13, \\ 2m + 2i & \text{for } j = 15, \end{cases}$$

$$h_j(v_iv_{i+1}) = \begin{cases} 5m - 1 - i & \text{for } j = 7, \\ 4m - 1 + i & \text{for } j = 8, 9, 11, 12, 15, \\ 2m + 2i & \text{for } j = 10, 14, \\ 3m + 2i & \text{for } j = 13, \end{cases}$$

and if $i = 1, 2, \ldots, m$ then

$$h_j(u_iv_i) = \begin{cases} 2m + \frac{i+1}{2} & \text{for } j = 7, 9, 11, 13 \text{ and } i \text{ odd}, \\ 3m - 2 + \frac{i}{2} & \text{for } j = 7, 9, 11, 13 \text{ and } i \text{ even}, \\ 3m + 1 - i & \text{for } j = 8, \\ 5m - 1 - i & \text{for } j = 10, \\ 2m + i & \text{for } j = 12, \\ 4m - 2 + i & \text{for } j = 14, \\ 2m - 1 + 2i & \text{for } j = 15. \end{cases}$$

One can see that for $j = 7, 8, 9, \ldots, 15$ every edge labeling h_j attains the values from the set $\{2m+1, 2m+2, \ldots, 5m-2\}$ and every labeling $h \cup h_j$ satisfies the properties of a total labeling of the ladder L_m. Then according to (2) and (4) we get

$$wt_{h_j}(C_4^{(i)}) = w_h + w_{h_j} = 8i + 2 + w_{h_j}. \tag{5}$$

For $j = 7$, $w_{h_7} = 14m - i - 2$ and $wt_{h_7}(C_4^{(i)}) = 14m + 7i$ for every $i = 1, 2, \ldots, m-1$. Under the labeling $h \cup h_7$ the C_4-weights of L_m create the arithmetic progression of the difference $d = 7$ and L_m is a super $(14m + 7, 7)$-C_4-antimagic.

For $j = 8$, $w_{h_8} = 13m$ and $wt_{h_8}(C_4^{(i)}) = 13m + 2 + 8i$ for every $i = 1, 2, \ldots, m-1$ and it proves that the total labeling $h \cup h_8$ is a super $(13m + 10, 8)$-C_4-antimagic.

For $j = 9$, $w_{h_9} = 13m - 2 + i$ and $wt_{h_9}(C_4^{(i)}) = 13m + 9i$ for every $i = 1, 2, \ldots, m-1$. It shows that the labeling $h \cup h_9$ is a super C_4-antimagic with the difference $d = 9$.

For $j = 10$, $w_{h_{10}} = 14m - 4 + 2i$ and $wt_{h_{10}}(C_4^{(i)}) = 14m - 2 + 10i$ for every $i = 1, 2, \ldots, m-1$. Under the labeling $h \cup h_{10}$ the C_4-weights of L_m form the arithmetic sequence with the difference $d = 10$ and L_m is a super $(14m + 8, 10)$-C_4-antimagic.

For $j = 11$, $w_{h_{11}} = 12m - 2 + 3i$ and $wt_{h_{11}}(C_4^{(i)}) = 12m + 11i$ for every $i = 1, 2, \ldots, m-1$ and it proves that the total labeling $h \cup h_{10}$ is a super $(12m + 11, 11)$-C_4-antimagic.

For $j = 12$, $w_{h_{12}} = 11m + 4i$ and $wt_{h_{12}}(C_4^{(i)}) = 11m + 2 + 12i$ for every $i = 1, 2, \ldots, m-1$ and it proves that the total labeling $h \cup h_{12}$ is a super $(11m + 14, 12)$-C_4-antimagic.

For $j = 13$, $w_{h_{13}} = 11m - 2 + 5i$ and $wt_{h_{13}}(C_4^{(i)}) = 11m + 13i$ for every $i = 1, 2, \ldots, m-1$. It shows that the labeling $h \cup h_{13}$ is a super C_4-antimagic with the difference $d = 13$.

For $j = 14$, $w_{h_{14}} = 12m - 4 + 6i$ and $wt_{h_{14}}(C_4^{(i)}) = 12m - 2 + 14i$ for every $i = 1, 2, \ldots, m - 1$. Under the labeling $h \cup h_{14}$ the C_4-weights of L_m form the arithmetic sequence with the difference $d = 14$ and L_m is a super $(12m + 12, 14)$-C_4-antimagic.

For $j = 15$, $w_{h_{15}} = 10m - 1 + 7i$ and $wt_{h_{15}}(C_4^{(i)}) = 10m + 1 + 15i$ for every $i = 1, 2, \ldots, m - 1$. It shows that the labeling $h \cup h_{15}$ is a super C_4-antimagic with the difference $d = 15$.

Thus we have arrived at the desired result. $\square$

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