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On Generalized High Order Derivatives of Nonsmooth Functions

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Abstract

In this paper, we proposed a Extension Definition to derive, simultaneously, the first, second and high order generalized derivatives for non-smooth functions, in which the involved functions are Riemann integrable but not necessarily locally Lipschitz or continuous. Indeed, we define a functional optimization problem corresponding to smooth functions where its optimal solutions are the first and second derivatives of these functions in a domain. Then by applying these functional optimization problems for non-smooth functions and using this method we obtain generalized first derivative (GFD) and generalized second derivative (GSD). Here, the optimization problem is approximated with a linear programming problem that by solving of which, we can obtain these derivatives, as simple as possible. We extend this approach for obtaining generalized high order derivatives (GHODs) of non-smooth functions, simultaneously. Finally, for efficiency of our approach some numerical examples have been presented.

Keywords

1. Introduction

A function is *smooth* (of the first order) if it is differentiable and its derivatives are continuous. The nth-order smoothness is defined analogously, that is, a function is nth-order smooth if its $(n-1)$th -derivatives are smooth. So, the infinite smoothness refers to continuous derivatives of all orders. From this perspective a non-smooth function only has a negative description—it lacks some degree of properties traditionally relied upon in analysis. One could get the impression that non-smooth optimization is a subject dedicated to overcoming handicaps which have to be faced in miscellaneous circumstances where mathematical structure might be poorer than what one would like, but this is far from right. Instead, non-smooth optimization typically deals with

highly structured problems, but problems which arise differently, or are modeled or cast differently, from ones for which many of the mainline numerical methods, involving gradient vectors and Hessian matrices, have been designed. Moreover, non-smooth analysis, which refers to differential analysis in the absence of differentiability and can be regarded as a subfield of nonlinear analysis, has grown rapidly in the past decades. In fact, in recent years, non-smooth analysis has come to play a vital role in functional analysis, optimization, mechanics, differential equations, etc.

Among those who have participated in its development are Clarke [1], Ioffe [2], Mordukhovich [3], Rockafellar [4], but many more have contributed as well. During the early 1960s there was a growing realization that a large number of optimization problems which appeared in applications involved minimization of non-differentiable functions. One of the important areas where such problems appear is optimum control. The subject of non-smooth analysis arises out of the need to develop a theory to deal with the minimization of non-smooth functions.

Recent research on non-smooth analysis mainly focuses on Lipschittz function. Properties of the generalized derivatives of Lipschtitz functions are summarized in the following result of Clarke, 1983 [1]. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be Lipschitz on a set U if there is a positive real number K such that

$$\left| f(y) - f(z) \right| \le K \|y - z\| \text{ for all } y, z \in U.$$

A function f that is Lipschitz in a neighborhood of a point x is not necessarily differentiable at x, but for locally Lipschitz functions the following expressions exist

$$f^+(x, v) := \limsup_{\lambda \downarrow 0} \frac{f(x + \lambda v) - f(x)}{\lambda}$$

and

$$f^-(x, v) := \liminf_{\lambda \downarrow 0} \frac{f(x + \lambda v) - f(x)}{\lambda}.$$

The symbols $f^+(x, v)$ and $f^-(x, v)$ denote the upper Dini and lower Dini directional derivatives of f at x in direction of v. Hence, we may consider any of these derivatives as a generalized derivative for locally Lipschitz functions. However they suffer from an important drawback. Both of them are in general not convex in the direction and simple examples (which are left to the reader) can be constructed to demonstrate this. Thus they lack the most important property of the directional derivative of a convex function. Clarke observed that in the definition of the upper Dini derivative if one moves the point x, *i.e.*, move through points that converge to x, then one can generate a generalized directional derivative which is sublinear in the direction. Thus we arrive at the definition of the Clarke generalized directional derivative.

Definition 1.1. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a locally Lipschitz function. Then the Clarke generalized directional derivative of f at x in direction of v is given by*

$$f^o(x, v) := \limsup_{\lambda \downarrow 0} \frac{f(x + \lambda v) - f(x)}{\lambda}.$$

Clarke also defined the following notion of generalized gradient or subdifferential [1].

Definition 1.2. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be locally Lipschitz. Then the Clarke generalized gradient or the subdifferential of f at x, denoted by* $\partial f(x)$, *is given by*

$$\partial f(x) = co \left\{ v \in \mathbb{R}^n : \exists \left(x^k \in D(f), x^k \to x, k \to \infty \right); v = \lim_{k \to \infty} \nabla f(x^k) \right\}.$$

Here $\nabla f(\cdot)$ *denotes the gradient of f;* $D(f)$ *denotes the set under which f is differentiable and co denotes the convex hull of a set. It is shown in* [5] *that the mapping* $\partial f(x)$ *is upper semicontinuous and bounded on every bounded set.*

However, one can realize that calculating the Clarke subdifferential from first principles is not always a simple task. In this manuscript, we develop the definition for GFD of non-smooth functions which will be seen as a refinement of what is given in [6], to derive GHODs of non-smooth functions, simultaneously.

The paper is organized as follows. Section 2 presents the basic definitions and facts needed in what follows. Our results are then stated and proved in Section 3. We develop the definition for GFD and GSD of non-smooth functions which will be seen as a refinement of what is given in theorems (2.1) and (2.7), to derive GHODs of non-smooth functions, simultaneously (Theorem 3.5). Section 4 presents some illustrative example of the results

of the paper.

2. GFD and GSD of Nonsmooth Functions

In this section, we present definitions and results concerning GFD and GSD, which are needed in the remainder of this paper. In order to get this approach some tools from nonsmooth analysis are used, specially generalized derivatives. Since the early 1960s, many different generalized derivatives have been proposed, for instance Rockafellar [4] [7], Clarke [1], Clarke *et al.* [8] and Mordukhovich [3] [9]-[12]. These papers and theirs results include some restrictions, for examples.

1) The function $f(\cdot)$ must be locally Lipschitz or convex.

2) We must know that the function $f(\cdot)$ is non-differentiable at a fixed point $x \in [a,b]$.

3) The generalized derivatives of $f(\cdot)$ on $[a,b]$ is a set, which either is empty or including several members.

4) The directional derivative is used to introduce generalized derivative.

5) The concepts limsup and liminf are applied to obtain the generalized derivative in which calculation of these is usually hard and complicated.

6) To obtain the second derivative, the gradient of the function should be computed which will be hard in some cases.

It is commonly recognized that these GDs are not practical and applicable for solving problems. We mainly use the new GD of Kamyad *et al.* [6] for nonsmooth functions. This kind of GD is particularly helpful and practical when dealing with nonsmooth continuous and discontinuous functions and does not have the above restrictions and difficulties. Let

$$V = \left\{ v_k(\cdot) : v_k(x) = \sin k\pi x, \ x \in [0,1], \ k = 1, 2, \cdots \right\}$$

Given a function $f(x)$ on $[0,1]$, consider the following functional optimization problem:

$$\text{Minimize} \quad J\left(g_1(\cdot)\right) = \sum_{k=1}^{\infty} \left| \int_0^1 v_k(x) g(x) \mathrm{d}x - \lambda_k \right|,$$

$$\text{subject to} \quad \int_{s_i-\delta}^{s_i+\delta} \left| f(x) - f(s_i) - (x - s_i) g(s_i) \right| \mathrm{d}x \leq \varepsilon \delta^2, \tag{1}$$

$$g(\cdot) \in C[0,1], \ s_i \in \left(\frac{i-1}{m}, \frac{i}{m} \right), \ i = 1, 2, \cdots, m$$

where $v_k(\cdot) \in V$,

$$\lambda_k = -\int_0^1 v_k'(x) f(x) \mathrm{d}x, \quad k = 1, 2, 3, \cdots$$

and ε, δ are sufficiently small positive numbers.

Theorem 2.1. (*Kamyad et al.*, 2011) *Let* $f(\cdot) \in C^1[0,1]$ *and* $g^*(\cdot) \in C[0,1]$ *be the optimal solution of the functional optimization problem* (1). *Then* $f'(\cdot) = g(\cdot)$.

Definition 2.2. (*Kamyad et al.*, 2011) *Let* $f(\cdot)$ *be a nonsmooth continuous function on the interval* $[0,1]$ *and* $g^*(\cdot)$ *be the optimal solution of the functional optimization problem* (1). *The generalized first derivative* (*GFD*) *of* $f(\cdot)$ *is denoted by* $\mathrm{GF}_d f(\cdot)$ *and defined as* $\mathrm{GF}_d f(\cdot) = g^*(\cdot)$.

Remark 2.3. *Note that if* $f(\cdot)$ *is a smooth function, then* $\mathrm{GF}_d f(\cdot)$ *as introduces in Definition* 2.2 *is equal to* $f'(\cdot)$. *Further, if* $f(\cdot)$ *is a nonsmooth function, then GFD of* $f(\cdot)$ *is an approximation for the first derivative of* $f(\cdot)$.

In what follow, the problem (1) is approximated as the following finite dimentional problem (Kamyad *et al.* [6]):

$$\text{Minimize} \quad J\left(g_1(\cdot)\right) = \sum_{k=1}^{N} \left| \int_0^1 v_k(x) g(x) \mathrm{d}x - \lambda_k \right|,$$

$$\text{subject to} \quad \int_{s_i-\delta}^{s_i+\delta} \left| f(x) - f(s_i) - (x - s_i) g(s_i) \right| \mathrm{d}x \leq \varepsilon \delta^2, \tag{2}$$

$$g(\cdot) \in C[0,1], \ s_i \in \left(\frac{i-1}{m}, \frac{i}{m} \right), \ i = 1, 2, \cdots, m$$

where *N* is a given large number. We assume that $g_i = g(s_i)$, $f_{i1} = f(s_i - \delta)$, $f_{i2} = f(s_i + \delta)$ and $f_i = f(s_i)$

for all $i = 1, 2, \cdots, m$. In addition, the authors choose the arbitrary points $s_i \in \left(\dfrac{i-1}{m}, \dfrac{i}{m} \right)$, $i = 1, 2, \cdots, m$.

By trapezoidal and midpoint integration rules, the last problem can be written as the following problem in which g_1, g_2, $\cdots$, g_m are its unknown variables:

$$\text{Minimize} \quad \sum_{k=1}^{N} \left| \delta \sum_{i=1}^{m} v_{ki} g_i - \lambda_k \right|,$$

$$\text{subject to} \quad \left| f_{i1} - f_i + \delta g_i \right| + \left| f_{i2} - f_i - \delta g_i \right| \leq \varepsilon \delta \qquad i = 1, 2, \cdots, m. \tag{3}$$

Lemma 2.4. *Let the pairs* $\left(v_i^*, u_i^* \right)$, $i = 1, 2, \cdots, m$ *be the optimal solution of the following linear programming* (*LP*) *problem*:

$$\text{Minimize} \quad \sum_{i=1}^{m} v_i,$$

$$\text{subject to} \quad v_i \geq u_i, \ v_i \geq -u_i, \ v_i \geq 0, \ u_i \in I.$$

Then u_i^*, $i = 1, 2, \cdots, m$ *are the optimal solutions of the following nonlinear programming problem* (*NLP*):

$$\text{Minimize}_{u \in I} \quad \sum_{i=1}^{m} |u_i|$$

where I is a compact set.

Proof. See [13].

Now, by Lemma 2.4, the problem (3) can be converted to the following equivalent LP problem (see [14] [15])

$$\text{Minimize} \quad \sum_{k=1}^{N} \mu_k,$$

$$\begin{aligned}
\text{subject to} \quad & -\mu_k + \delta \sum_{i=0}^{m} v_{ki} g_i \leq \lambda_k, \quad k = 1, 2, \cdots, N, \\
& -\mu_k - \delta \sum_{i=0}^{m} v_{ki} g_i \leq -\lambda_k, \quad k = 1, 2, \cdots, N, \\
& (u_i + q_i) + (w_i + z_i) \leq \varepsilon \delta, \quad i = 1, 2, \cdots, m, \\
& u_i - q_i - \delta g_i = f_{i1} - f_i, \quad i = 1, 2, \cdots, m, \\
& w_i - z_i + \delta g_i = f_{i2} - f_i, \quad i = 1, 2, \cdots, m, \\
& w_i, z_i, u_i, q_i, \mu_k \geq 0, \quad k = 1, 2, \cdots, N.
\end{aligned} \tag{4}$$

where g_i, μ_k and w_i, z_i, u_i, q_i are decision variables of the problem for $i = 1, 2, \cdots, m$ and $k = 1, \cdots, N$ and

$$\lambda_k = -\int_0^1 v_k'(x) f(x) \, \mathrm{d}x, \quad k = 1, 2, 3, \cdots, N$$

and ε, δ are positive sufficiently small numbers.

Remark 2.5. *Note that* ε *and* δ *are sufficiently small numbers and the points* $s_i \in \left(\dfrac{i-1}{m}, \dfrac{i}{m} \right)$ *can be chosen as arbitrary numbers for* $i = 1, 2, \cdots, m$.

Remark 2.6. *Let* g_i^* *be the optimal solution of problem* (1) *for* $i = 1, 2, \cdots, m$. *Then* $\text{GF}_d f(s_i) = g_i^*$ *for* $i = 1, 2, \cdots, m$.

Based on GFD of nonsmooth functions, Kamyad *et al.* [6] introduced an optimization problem similar to the problem (1) that its optimal solution is the second order derivative of smooth functions on an interval. To attain this goal they define

$$V = \left\{ v_k(\cdot) : v_k(x) = 1 - \cos(2k\pi x) : \ x \in [0,1], k \in \mathbb{N} \right\}.$$

In addition, suppose $\varepsilon > 0$ and $\delta > 0$ are two sufficiently small numbers and $m \in \mathbb{N}$. For given continuous function $f(x)$ on $[0,1]$, defined the following functional optimization problem:

$$\text{Minimize} \quad I\big(g_1(\cdot)\big) = \sum_{k=1}^{\infty} \left| \int_0^1 v_k(x) g(x) \mathrm{d}x - \lambda_k \right|,$$

$$\text{subject to} \quad \int_{s_i-\delta}^{s_i+\delta} \left| h(x) - h(s_i) - (x - s_i) g(s_i) \right| \mathrm{d}x \leq \varepsilon \delta^2, \tag{5}$$

$$g(\cdot) \in C[0,1], \quad s_i \in \left(\frac{i-1}{m}, \frac{i}{m} \right), \quad i = 1, 2, \cdots, m$$

where $v_k(\cdot) \in V$ for all $k = 1, 2, 3, \cdots$ and $\lambda_k = \int_0^1 v_k(x) f''(x) \mathrm{d}x$. Also if $f(\cdot) \in C^1[0,1]$ then $h(\cdot) = f'(\cdot)$ and otherwise $h(\cdot) = \mathrm{GF}_d f(\cdot)$ where $\mathrm{GF}_d f(\cdot)$ is the GFD of $f(x)$ (see [6]).

Theorem 2.7. *Let* $f(\cdot) \in C^2[0,1]$ *and* $g^*(\cdot) \in C[0,1]$ *is the optimal solution of functional optimization problem* (5). *Then* $g^*(\cdot) = f''(\cdot)$.

Proof. See [16].

Definition 2.8. *Let* $f(\cdot)$ *be a continuous nonsmooth function on interval* $[0,1]$ *and* $g^*(\cdot)$ *be the optimal solution of the functional optimization problem* (5). *We denote the GSD of* $f(\cdot)$ *by* $\mathrm{GS}_d f(\cdot)$ *and it is defined as* $\mathrm{GS}_d f(\cdot) = g^*(\cdot)$.

Note that if $f(\cdot)$ is a smooth function then the $\mathrm{GS}_d f(\cdot)$ is $f''(\cdot)$. Also if $f(\cdot)$ is a nonsmooth function then GSD of $f(\cdot)$ is an approximation for second order derivative of $f(\cdot)$. Thus, according to the above mentioned statements for GFD and generalized results, problem (5) is converted to the following equivalent linear programming problem:

$$\text{Minimize} \quad \sum_{k=1}^{N} \mu_k,$$

$$\text{subject to} \quad -\mu_k + \delta \sum_{i=0}^{m} v_{ki} g_i \leq \lambda_k, \quad k = 1, 2, \cdots, N$$

$$-\mu_k - \delta \sum_{i=0}^{m} v_{ki} g_i \leq -\lambda_k, \quad k = 1, 2, \cdots, N$$

$$(u_i + q_i) + (w_i + z_i) \leq \varepsilon \delta, \quad i = 1, 2, \cdots, m \tag{6}$$

$$u_i - q_i - \delta g_i = h_{i1} - h_i, \quad i = 1, 2, \cdots, m$$

$$w_i - z_i + \delta g_i = h_{i2} - h_i, \quad i = 1, 2, \cdots, m$$

$$w_i, z_i, u_i, q_i, \mu_k \geq 0, \quad k = 1, 2, \cdots, N$$

where g_i, μ_k, w_i, z_i, u_i, q_i are decision variables of the problem for $i = 1, 2, \cdots, m$ and $k = 1, \cdots, N$ and $\lambda_k = \int_0^1 v_k(x) f''(x) \mathrm{d}x$ for $k = 1, 2, 3, \cdots, N$ and ε, δ are sufficiently small positive numbers.

Remark 2.9. *Let* g_i^* *be the optimal solution of problem* (6) *for* $i = 1, 2, \cdots, m$. *Then we have* $\mathrm{GS}_d f(s_i) = g_i^*$ *for all* $i = 1, 2, \cdots, m$.

3. Main Approach

In this section, we propose a method based on GD of Kamyad *et al*. to obtain simultaneously GFD and GSD for smooth and nonsmooth functions in which the involved functions are Riemann integrable.

Theorem 3.1. *Let* $f(\cdot), g_1(\cdot), g_2(\cdot) \in C[0,1]$ *and*

$$\int_0^1 \big(v'(x) f(x) + v(x) g_1(x)\big) \mathrm{d}x = 0,$$

$$\int_0^1 \big(v''(x) f(x) + 2v(x) g_1(x) + v(x) g_2(x)\big) \mathrm{d}x = 0.$$

for any $v(\cdot) \in C^2[0,1]$ *where* $v(0) = v(1) = 0$, $v'(0) = v'(1) = 0$. *Then we have* $f(\cdot) \in C^2[0,1]$ *and* $f'(\cdot) = g_1(\cdot)$, $g_1'(\cdot) = g_2(\cdot)$.

Proof. By Theorem 2.2 in [6] we have $f'(\cdot) = g(\cdot)$. Now we use integration by parts and the first condition.

$$\int_0^1 v''(x) f(x) \mathrm{d}x = \big[v'(x) f(x)(x) \big]_0^1 - \int_0^1 v'(x) g_1(x) \mathrm{d}x = -\int_0^1 v'(x) g_1(x) \mathrm{d}x \tag{7}$$

Since by assumption

$$\int_0^1 v'(x) g_1(x) \mathrm{d}x = \left[v'(x) G_1(x) \right]_0^1 - \int_0^1 v''(x) G_1(x) \mathrm{d}x = -\int_0^1 v''(x) G_1(x) \mathrm{d}x$$

and

$$\int_0^1 v(x) g_2(x) \mathrm{d}x = \left[v(x) G_2(x) \right]_0^1 - \int_0^1 v'(x) G_2(x) \mathrm{d}x = -\int_0^1 v'(x) G_2(x) \mathrm{d}x.$$

Again by using integration by parts and rule for $\int_0^1 v'(x) G_2(x) \mathrm{d}x$ we have:

$$\int_0^1 v'(x) G_2(x) \mathrm{d}x = \left[v'(x) G_3(x) \right]_0^1 - \int_0^1 v''(x) G_3(x) \mathrm{d}x = -\int_0^1 v''(x) G_3(x) \mathrm{d}x,$$

where $G_1(x) = \int_0^x g_1(t) \mathrm{d}t$, $G_2(x) = \int_0^x g_2(t) \mathrm{d}t$ and $G_3(x) = \int_0^x G_2(t) \mathrm{d}t$ for each $x \in [0,1]$. Since by assumption $\int_0^1 \left(v''(x) f(x) + 2v'(x) g_1(x) + v(x) g_2(x) \right) \mathrm{d}x = 0$, (7) yields $\int_0^1 \left(v(x) g_2(x) + v'(x) g_1(x) \right) \mathrm{d}x = 0$ so that

$$\int_0^1 v''(x) \left(G_3(x) - G_1(x) \right) \mathrm{d}x = 0.$$

By Lemma 2.1 of [6] it follows that $g_1'(\cdot) = g_2(\cdot)$. Thus, by Theorem 2.2 of [6], we conclude that $f''(\cdot) = g_2(\cdot)$.

Corollary 3.2. *Suppose that the conditions of Theorem 3.1 hold. Then $f(\cdot) \in C^2[0,1]$ and $f''(\cdot) = g_2(\cdot)$.*
Let

$$V = \left\{ v_k(\cdot) : v_k(x) = 1 - \cos(2k\pi x), \ x \in [0,1], k \in \mathbb{N} \right\}.$$

Then $v_k(0) = v_k(1) = 0$ and $v_k'(0) = v_k'(1) = 0$ for all $v_k(\cdot) \in V$. We note that V is a total set in $U = \left\{ v(\cdot) \in C^2[0,1] : v(0) = v(1) = v'(0) = v'(1) = 0 \right\}$. Hence, for any $v(\cdot) \in U$, there exist coefficients $c_0, c_1, \cdots \in \mathbb{R}$ such that

$$v(x) = \sum_{k=1}^\infty c_k v_k(x) : v_k(\cdot) \in V, \ x \in [0,1].$$

Theorem 3.3. *Let $f(\cdot)$, $g_1(\cdot)$, $g_2(\cdot) \in C^2[0,1]$ and*

$$\int_0^1 \left(v_k'(x) f(x) + v_k(x) g_1(x) \right) \mathrm{d}x = 0,$$

$$\int_0^1 \left(v_k''(x) f(x) + 2v_k'(x) g_1(x) + v_k(x) g_2(x) \right) \mathrm{d}x = 0.$$

for any $v_k(\cdot) \in V$. Then we have $f(\cdot) \in C^2[0,1]$ and

$$f'(\cdot) = g_1(\cdot), \quad f''(\cdot) = g_2(\cdot).$$

Theorem 3.4. *Let $\varepsilon > 0$ be a small number, $f(\cdot) \in C^2[0,1]$ and $m \in \mathbb{N}$. Then there exist $\delta > 0$ such that for all $s_i \in \left(\dfrac{i-1}{m}, \dfrac{i}{m} \right)$ and $i = 1, 2, \cdots, m$,*

$$\int_{s_i - \delta}^{s_i + \delta} \left| f(x) - f(s_i) - (x - s_i) f'(s_i) - \frac{1}{2}(x - s_i)^2 f''(s_i) \right| \mathrm{d}x \leq 2\varepsilon\delta^2.$$

Proof. The result is clear by Theorem 2.4 of [6] and the integral properties.

Let $\varepsilon > 0$ and $\delta > 0$ be two sufficiently small given numbers and $m \in \mathbb{N}$. For a given function $f(x)$ on $[0,1]$, the following functional optimization problem defined the Extension generalized derivative (EGD):

$$\text{Minimize} \quad L\left(g_1(\cdot), g_2(\cdot) \right) = \sum_{k=1}^N \left| \int_0^1 \left(v_k(x) g_2(x)(x) + 2v_k'(x) g_1(x) + v_k''(x) f(x) \right) \mathrm{d}x \right|$$

$$\text{subject to} \quad \int_{s_i - \delta}^{s_i + \delta} \left| f(x) - f(s_i) - (x - s_i) g_1(s_i) \right| \mathrm{d}x \leq \varepsilon\delta^2,$$

$$\int_{s_i - \delta}^{s_i + \delta} \left| f(x) - f(s_i) - (x - s_i) g_1(s_i) - \frac{1}{2}(x - s_i)^2 g_2(s_i) \right| \mathrm{d}x \leq 2\varepsilon\delta^2, \tag{8}$$

$$g_1(\cdot), g_2(\cdot) \in C^1[0,1], s_i \in \left(\frac{i-1}{m}, \frac{i}{m} \right), \quad i = 1, 2, \cdots, m.$$

where $v_k(\cdot) \in V$ for all $k = 1, 2, 3, \cdots$.

Theorem 3.5. Let $f(\cdot) \in C^2[0,1]$ and $g_1^*(\cdot)$, $g_2^*(\cdot) \in C^1[0,1]$ be the optimal solution of the functional optimization problem (8). Then $f'(\cdot) = g_1^*(\cdot)$, $f''(\cdot) = g_2^*(\cdot)$.

Proof. The result follows from Theorems 3.3, 3.4.

Definition 3.6. Let $f(\cdot)$ be a nonsmooth continuous function on the interval $[0,1]$ and $g_1^*(\cdot)$, $g_2^*(\cdot)$ be the optimal solutions of the functional optimization problem (8). The Extension generalized first and second derivatives (EGDs) of $f(\cdot)$ is denoted by $\mathrm{GF}_d f(\cdot)$ and $\mathrm{GS}_d f(\cdot)$, respectively, and defined as $\mathrm{GF}_d f(\cdot) = g_1^*(\cdot)$ and $\mathrm{GS}_d f(\cdot) = g_2^*(\cdot)$.

Definition 3.7. Let $f(\cdot)$ be a nonsmooth Lebesgue integrable function on the interval $[0,1]$ and $g_1^*(\cdot)$, $g_2^*(\cdot)$, be the Lebsgue integrable functions optimal solutions of the functional optimization problem (8). The EGDs of $f(\cdot)$ is denoted by $\mathrm{GF}_d f(\cdot)$ and $\mathrm{GS}_d f(\cdot)$, respectivily, and defined as $\mathrm{GF}_d f(\cdot) = g_1^*(\cdot)$ and $\mathrm{GS}_d f(\cdot) = g_2^*(\cdot)$.

Note that if $f(\cdot)$ is a smooth function, then $\mathrm{GF}_d f(\cdot)$ in Definition 3.6 is equal to $f'(\cdot)$ and $\mathrm{GS}_d f(\cdot)$ is equal to $f''(\cdot)$. Further, if $f(\cdot)$ is a nonsmooth function, then GFD of $f(\cdot)$ is an approximation for the first derivative and GSD of $f(\cdot)$ is an approximation for the second derivative of $f(\cdot)$.

We approximate the obtained generalized derivatives of nonsmooth function $f(\cdot)$ with Fourier series

$$g_1(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos(k\pi x) + b_k \sin(k\pi x) \right),$$

$$g_2(x) = \frac{c_0}{2} + \sum_{k=1}^{\infty} \left(c_k \cos(k\pi x) + d_k \sin(k\pi x) \right),$$

where the coefficients a_k, b_k, c_k, d_k satisfy the following relations:

$$a_k = \int_{-1}^{1} \cos(k\pi x) g_1(x) \mathrm{d}x, \quad b_k = \int_{-1}^{1} \sin(k\pi x) g_1(x) \mathrm{d}x,$$

$$c_k = \int_{-1}^{1} \cos(k\pi x) g_2(x) \mathrm{d}x, \quad d_k = \int_{-1}^{1} \sin(k\pi x) g_2(x) \mathrm{d}x.$$

$k = 1, 2, \cdots$. From Fourier analysis [17], we have $\lim_{k \to \infty} a_k = 0$, $\lim_{k \to \infty} b_k = 0$, $\lim_{k \to \infty} c_k = 0$ and $\lim_{k \to \infty} d_k = 0$. Hence there exists $N \in \mathbb{N}$ such that $a_k \approx 0$, $b_k \approx 0$, $c_k \approx 0$ and $d_k \approx 0$ for all $k > N$.

Note that, we can approximate the generalized derivatives of nonsmooth function $f(\cdot)$ with polynomials $\sum_{n=0}^{M} a_n x^n$ of degree M. For this purpose put

$$\mathrm{GF}_d f(s_i) \cong g_1(s_i) = \sum_{n=0}^{M} a_n s_i^n$$

and

$$\mathrm{GS}_d f(s_i) \cong g_2(s_i) = \sum_{n=0}^{M} b_n s_i^n.$$

Hence, in what follow, by using the methods of [6], we will convert the problem (8) to the corresponding finite dimensional problem,

$$\text{Minimize} \quad L(g_1(\cdot), g_2(\cdot)) = \sum_{k=1}^{N} \left| \int_0^1 \left(v_k(x) g_2(x)(x) + 2v_k'(x) g_1(x) \right) \mathrm{d}x \right|$$

$$\text{subject to} \quad \int_{s_i - \delta}^{s_i + \delta} \left| f(x) - f(s_i) - (x - s_i) g_1(s_i) \right| \mathrm{d}x \leq \varepsilon \delta^2,$$

$$\int_{s_i - \delta}^{s_i + \delta} \left| f(x) - f(s_i) - (x - s_i) g_1(s_i) - \frac{1}{2}(x - s_i)^2 g_2(s_i) \right| \mathrm{d}x \leq 2\varepsilon \delta^2, \tag{9}$$

$$g_1(\cdot), g_2(\cdot) \in C^1[0,1], s_i \in \left(\frac{i-1}{m}, \frac{i}{m} \right), \quad i = 1, 2, \cdots, m,$$

$$\lambda_k = -\int_0^1 v_k''(x) f(x) \mathrm{d}x,$$

where N is a given big number. Let $f_i = f(s_i)$, $f_{i1} = f(s_i - \delta)$, $f_{i2} = f(s_i + \delta)$, $g_{2i} = g_2(s_i)$, $g_{1i} = g_1(s_i)$ and $v_{ki} = v_k(s_i)$ for $i = 1, 2, \cdots, m$ and $s_i = \left(\frac{i-1}{m}, \frac{i}{m} \right)$. Then by trapezoidal and midpoint integration rules,

problem 9 can be written as the following problem:

$$\text{Minimize} \quad L\big(g_1(\cdot), g_2(\cdot)\big) = \sum_{k=1}^{N} \left| \delta \sum_{i=1}^{m} \big(v_{ki}g_{2i} + 2v'_{ki}g_{1i}\big) - \lambda_k \right|$$

$$\text{subject to} \quad \left| f_{i1} - f_i + \delta g_{1i} \right| + \left| f_{i2} - f_i - \delta g_{1i} \right| \le \varepsilon\delta, \tag{10}$$

$$\left| f_{i1} - f_i + \delta g_{1i} - \frac{1}{2}\delta^2 g_{2i} \right| + \left| f_{i2} - f_i - \delta g_{1i} - \frac{1}{2}\delta^2 g_{2i} \right| \le 2\varepsilon\delta, \quad i = 1, 2, \cdots, m.$$

Now, problem (9) can be converted to the following equivalent linear programming problem (see [14] [15]):

$$\text{Minimize} \quad \sum_{k=1}^{N} \mu_k,$$

$$\text{subject to} \quad -\mu_k + \delta \sum_{i=0}^{m} \big(v_{ki}g_{2i} + 2v'_{ki}g_{1i}\big) \le \lambda_k,$$

$$-\mu_k - \delta \sum_{i=0}^{m} \big(v_{ki}g_{2i} + 2v'_{ki}g_{1i}\big) \le -\lambda_k,$$

$$(\tau_i + \rho_i) + (\sigma_i + \gamma_i) \le \varepsilon\delta,$$

$$\tau_i - \rho_i - \delta g_{1i} = f_{i1} - f_i, \tag{11}$$

$$\sigma_i - \gamma_i + \delta g_{1i} = f_{i2} - f_i,$$

$$(u_i + q_i) + (w_i + z_i) \le 2\varepsilon\delta,$$

$$u_i - q_i - \delta g_{1i} + \frac{1}{2}\delta^2 g_{2i} = f_{i1} - f_i,$$

$$w_i - z_i + \delta g_{1i} + \frac{1}{2}\delta^2 g_{2i} = f_{i2} - f_i,$$

$$\sigma_i, \gamma_i, \rho_i, \tau_i, w_i, z_i, u_i, q_i, \mu_k \ge 0,$$

where g_{1i}, g_{2i}, μ_k for $k = 1, 2, \cdots, N$ and w_i, z_i, u_i, q_i, τ_i, ρ_i, γ_i, σ_i for $i = 1, 2, \cdots, m$ are decision variables of the problem (11) and

$$\lambda_k = -\delta \sum_{i=1}^{m} v''_k(s_i) f(s_i) \mathrm{d}x, \quad k = 1, 2, 3, \cdots, N$$

for $i = 1, \cdots, m$ and ε, δ are sufficiently small positive numbers.

Remark 3.8. *Let* $f(\cdot) \in C^n[0,1]$ *and* $v(0) = v(1) = v'(0) = v'(1) = \cdots = v^{(n-1)}(0) = v^{(n-1)}(1) = 0$. *Then, by previous EGD and using the above method and Taylor expansion of order n, simultaneously, we calculate all derivatives up to nth order of* $f(\cdot)$.

Remark 3.9. *By looking at the definition of EGD, for a function as* $f(x) = \sum_{l=1}^{L} f_l(x_l)$ *where* $x = (x_1, x_2, \cdots, x_n)$, *we have* $\partial_{x_k} f(x) = \partial f_k(x_k)$ *for all* $k = 1, 2, \cdots, L$. *So, to obtain the EGD of the multi-variable function* $f(x)$ *with respect to* x_k, *we employ the LP problem* (11) *and gain the generalized first and second derivatives of one-variable function* $f_k(x_k)$.

4. Numerical Results

In this stage, we have found the GFD and GSD of smooth and nonsmooth functions in several examples using problem (11). Here we assume that $\varepsilon = \delta = 0.01$, $N = 20$, $m = 99$ and $s_i = 0.01i$ for all $i = 1, 2, \cdots, 99$. The problem (11) is solved for functions in these examples using simplex method in MATLAB software. Attend that in our approach points in $[0,1]$ are selected arbitrarily, and with selection very of these points, we can cover this interval. In this examples, $g1(x)$ shown the GFD and $g2(x)$ shown the GSD of $f(x)$.

Example 4.1. *Consider the nonsmooth function* $f(x) = |x| \mathrm{e}^{-|x|}$, $x \in [-1,1]$ *is illustrated in Figure 1. Figure 2 shows the GFD of* $f(\cdot)$, *which is the optimal solution of optimization problem* (4).

Example 4.2. *Consider the smooth function* $f(x) = \mathrm{e}^x \sin^2(x)$ *on interval* $[0,1]$. *The function* $f(x)$ *is illustrated in Figure 3. Figure 4 shows the GSD of* $f(x)$ *which has been obtained by using the problem* (6).
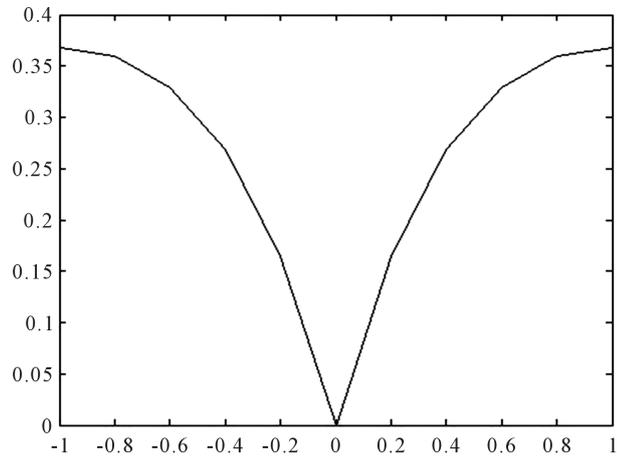
Figure 1. $f(x)$ of Example 4.1.
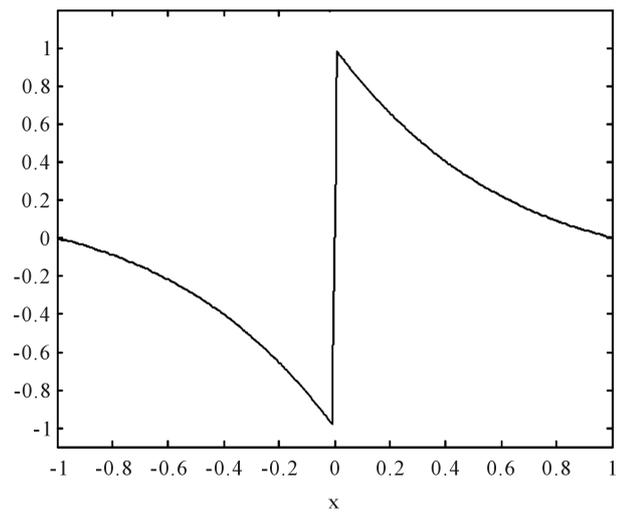
Figure 2. GFD of Example 4.1 by using the problem (4).
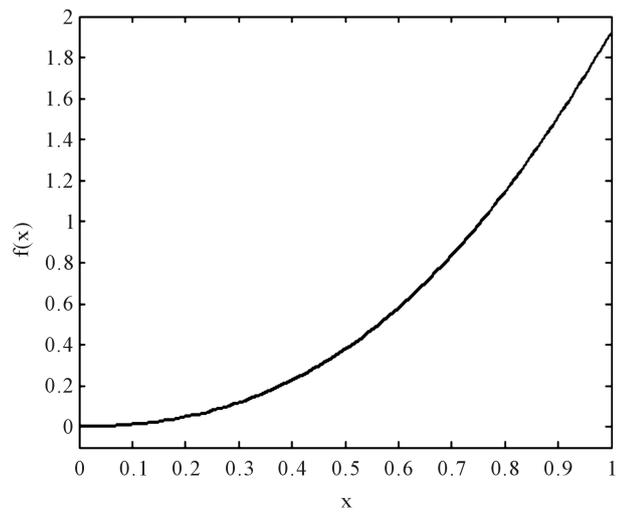
Figure 3. $f(x)$ of Example 4.2.

Example 4.3. *Consider the nonsmooth function* $f(x) = \left\| 2x - 1 \right| - 0.5 \right|$ *on* $[0,1]$. *This function is not differentiable in* $x = 0.25, 0.5, 0.75$, *and according to the problem* (11), *GFD and GSD of* $f(x)$ *has been shown in* **Figure 5**.

Example 4.4. *Consider the nonsmooth function* $f(x) = \sum_{i=1}^{10} \left| x - \dfrac{i}{10} \right|$ *on* $[0,1]$. *This function is not differentiable in* $x_i = \dfrac{i}{10}$. *This function and the GED of this function are shown in* **Figure 6**.

Example 4.5. *Consider function* $f(x) = \left| \cos(5\pi x) \right|$ *on interval* $(0,1)$ *which is a non-differentiable function in points* $x_i = \dfrac{2i-1}{10}$ *for any* $i = 1, 2, \cdots, 5$. *The function* $f(x) = \left| \cos(5\pi x) \right|$ *and the GED of this function are shown in* **Figure 7**.

5. Conclusions

We propose a strategy for approximating generalized first and second derivatives for a nonsmooth function. We assume that the function of interest which has satisfied, is different from what is usually found in the literature,
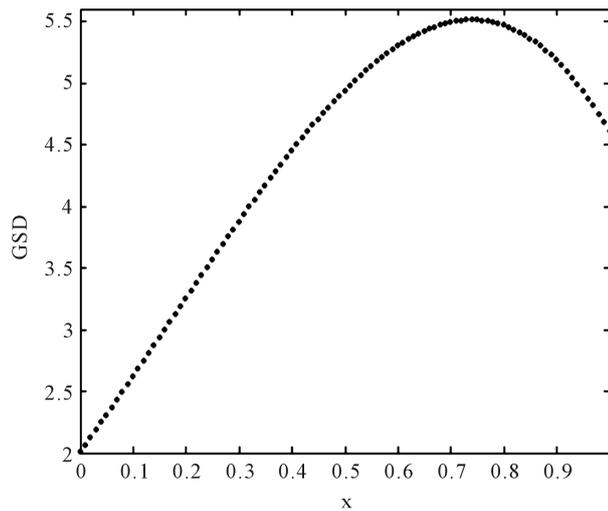
Figure 4. GSD of Example 4.2 by using the problem (6).
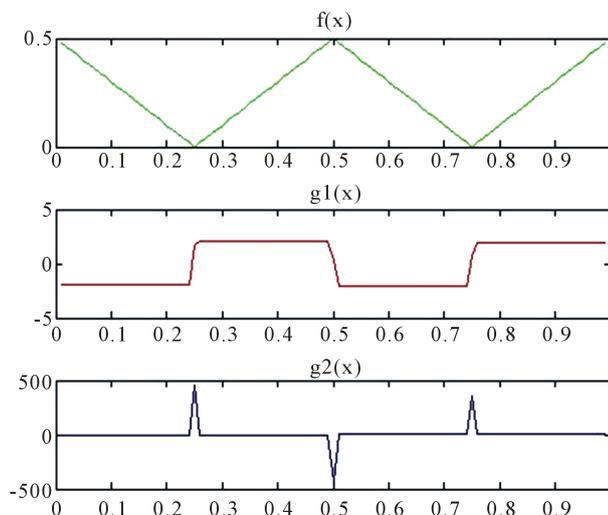
Figure 5. EGD of Example 4.3 by using the problem (11).

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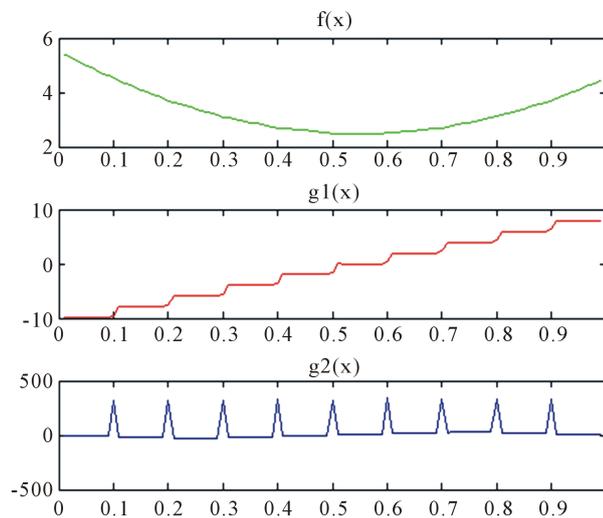
Figure 6. EGD of Example 4.4 by using the problem (11).
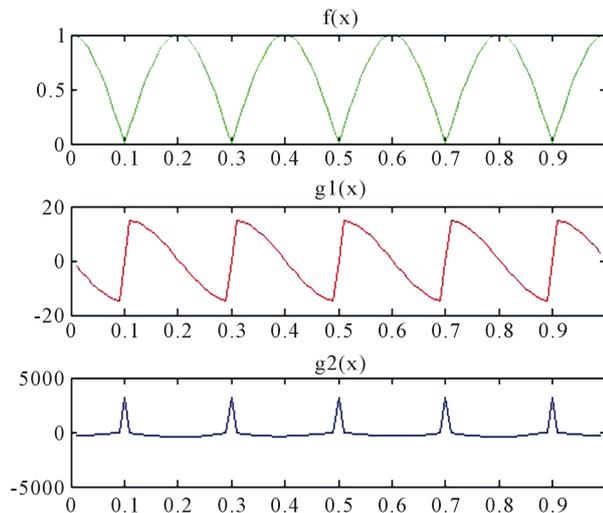
Figure 7. EGD of Example 4.5 by using the problem (11).

since it is not required to be locally Lipschitz or convex. These generalized derivatives are computed by solving a linear programming problem, whose solution provides simultaneously the first and second order derivatives. The advantages of our generalized derivatives with respect to the other approaches except simplicity and practically are as follows:

1) The generalized derivative of a non-smooth function by our approach does not depend on the non-smoothness points of function. Thus we can use this GD for many cases that we do not know the points of non-differrentiability of the function.

2) The generalized derivative of non-smooth functions by our approach gives a good global approximate derivative as on the domain of functions, whereas in the other approaches the GDs are calculated in one given point.

3) The generalized derivative by our approach is defined for non-smooth piecewise continuous functions, whereas the other approaches are defined usually for locally Lipschitz or convex functions.

4) Our approach simultaneously gives first and second derivatives of nonsmooth function plus second derivative obtained directly from the primary function without need to compute the first derivative.

5) The generalized derivative by our approach is valid for functions that are only integrable, where continuity and locally Lipschitz need not be assumed.

In deed we show that if $f:[a,b] \to \mathbb{R}$ is continuous, and if $\varphi:[a,b] \to \mathbb{R}$ is a smooth function that vanishes at the end points of the interval, then $\int_a^b f'(x)\varphi(x)\mathrm{d}x = -\int_a^b f(x)\varphi'(x)\mathrm{d}x$. In the following form, where continuity need not be assumed: if $f:[a,b] \to \mathbb{R}$ is an integrable function such that $\int_a^b g(x)\varphi(x)\mathrm{d}x = -\int_a^b f(x)\varphi'(x)\mathrm{d}x$ for some integrable $g:[a,b] \to \mathbb{R}$ and for all infinitely many times differentiable, or smooth, functions $\varphi:[a,b] \to \mathbb{R}$ such that $\varphi(a) = 0 = \varphi(b)$, then f agrees almost everywhere with an absolutely continuous function; moreover $f' = g$ almost everywhere in this case.

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